CBSE NCERT Solutions for Class 12 Maths Chapter 09

Exercise 9.1

1. Determine order and degree (if defined) of differential equation $\frac{d^4y}{dx^4} + \sin(y''') = 0$ **[1 Mark]**

Solution:

Given differential equation is $\frac{d^4y}{dx^4} + \sin(y''') = 0$

We know that the highest order derivative present in the differential equation is 4. Therefore, its order is four.
[1/2 Mark]

The given differential equation is not a polynomial equation in its derivatives. Hence, its degree is not defined. **[1/2 Mark]**

2. Determine order and degree (if defined) of differential equation $y' + 5y = 0$ **[1 Mark]**

Solution:

The given differential equation is $y' + 5y = 0$

The highest order derivative present in the differential equation is y'.

Therefore, its order is one. **[1/2 Mark]**

It is a polynomial equation in its derivatives and the highest power raised to y' is 1. **[1/2 Mark]**

Hence, its degree is one.

3. Determine order and degree (if defined) of differential equation $\left(\frac{ds}{dt}\right)^4 + 3s\frac{d^2s}{dt^2} = 0$ **[1 Mark]**

Solution:

Given differential equation is $\left(\frac{ds}{dt}\right)^4 + 3\frac{d^2s}{dt^2} = 0$

The highest order derivative present in the given differential equation is $\frac{d^2s}{dt^2}$. Therefore, its order is two.

[1/2 Mark]

It is a polynomial equation in $\frac{d^2s}{dt^2}$. and $\frac{ds}{dt}$ and the power raised to $\frac{d^2s}{dt^2}$ is 1. **[1/2 Mark]**

Hence, its degree is one.

4. Determine order and degree (if defined) of differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$ **[1 Mark]**

Solution:

Given differential equation is $\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$

The highest order derivative present in the given differential equation is $\frac{d^2y}{dx^2}$. Therefore, its order is 2.

[1/2 Mark]

The given differential equation is not a polynomial equation in its derivatives. Hence, its degree is not defined. **[1/2 Mark]**

5. Determine order and degree (if defined) of differential equation $\frac{d^2y}{dx^2} = \cos 3x + \sin 3x$ **[1 Mark]**

Solution:

Given differential equation is $\frac{d^2y}{dx^2} = \cos 3x + \sin 3x$

$$\Rightarrow \frac{d^2y}{dx^2} - \cos 3x - \sin 3x = 0$$

The highest order derivative present in the differential equation is $\frac{d^2y}{dx^2}$. Therefore, its order is two.

[1/2 Mark]

It is a polynomial equation in $\frac{d^2y}{dx^2}$ and the power raised to $\frac{d^2y}{dx^2}$ is 1. **[1/2 Mark]**

Hence, its degree is one.

6. Determine order and degree (if defined) of differential equation $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$ [1 Mark]

Solution:

Given differential equation is $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$

The highest order derivative present in the differential equation is y'''.

Therefore, its order is three. **[1/2 Mark]**

The given differential equation is a polynomial equation in y''', y'', and y'.

The highest power raised to y''' is 2. Hence, its degree is 2. **[1/2 Mark]**

7. Determine order and degree (if defined) of differential equation $y''' + 2y'' + y' = 0$ **[1 Mark]**

Solution:

Given differential equation is $y''' + 2y'' + y' = 0$

The highest order derivative present in the differential equation is y'''.

Therefore, its order is three. **[1/2 Mark]**

It is a polynomial equation in y''', y'', and y' and the highest power raised to y''' is 1.

Hence, its degree is 1. [$\frac{1}{2}$ Mark]

8. Determine order and degree (if defined) of differential equation $y' + y = e^x$ **[1 Mark]**

Solution:

Given differential equation is $y' + y = e^x$

$\Rightarrow y' + y - e^x = 0$

The highest order derivative present in the differential equation is y'.

Therefore, its order is one. **[1/2 Mark]**

The given differential equation is a polynomial equation in y' and the highest power raised to y' is one. Hence, its degree is one. **[1/2 Mark]**

9. Determine order and degree (if defined) of differential equation $y'' + (y')^2 + 2y = 0$ **[1 Mark]**

Solution:

Given differential equation is $y'' + (y')^2 + 2y = 0$

The highest order derivative present in the differential equation is y''.

Therefore, its order is two. **[1/2 Mark]**

The given differential equation is a polynomial equation in y'' and y' the highest power raised to y'' is one.

[1/2 Mark]

Hence, its degree is one.

10. Determine order and degree (if defined) of differential equation $y'' + 2y' + \sin y = 0$ **[1 Mark]**

Solution:

Given differential equation is $y'' + 2y' + \sin y = 0$

The highest order derivative present in the differential equation is y''.

Therefore, its order is two. **[1/2 Mark]**

This is a polynomial equation in y'' and y' and the highest power raised to y'' is one.

Hence, its degree is one. **[1/2 Mark]**

11. The degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0$ is **[1 Mark]**

A) 3

B) 2

C) 1

D) not defined

Solution:

Given differential equation is $\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0$

The given differential equation is not a polynomial equation in its derivatives. Therefore, its degree is not defined. **[1 Mark]**

Hence, the correct answer is D.

12. The order of the differential equation $2x^2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0$ is **[1 Mark]**

A) 2

B) 1

C) 0

D) not defined

Solution:

Given differential equation is $2x^2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0$

The highest order derivative present in the given differential equation is $\frac{d^2y}{dx^2}$.

Therefore, its order is two. **[1 Mark]**

Hence, the correct answer is A.

Exercise 9.2

In each of the Exercises 1 to 10 verify that the given functions (explicit or implicit) is a solution of the corresponding differential equation:

1. $y = e^x + 1 : y'' - y' = 0$ **[2 Marks]**

Solution:

Given equation is $y = e^x + 1$

Differentiating both the sides of the given equation with respect to x, we get:

$$\frac{dy}{dx} = \frac{d}{dx}(e^x + 1)$$

$$\Rightarrow y' = e^x \ldots (1)$$ **[1 Mark]**

Now, differentiating equation (1) with respect to x, we get:

$$\frac{d}{dx}(y') = \frac{d}{dx}(e^x)$$

$$\Rightarrow y'' = e^x$$

Substituting the values of y' and y'' in the given differential equation, we get the L.H.S. as:

$$y'' - y' = e^x - e^x = 0 = \text{R.H.S.}$$

Thus, the given function is the solution of the corresponding differential equation. **[1 Mark]**

2. $y = x^2 + 2x + C : y' - 2x - 2 -= 0$
[2 Marks]

Solution:

Given equation is $y = x^3 + 2x + C$

Differentiating both sides of this equation with respect to x we get:

$$y' = \frac{d}{dx}(x^2 + 2x + C)$$

$\Rightarrow y' = 2x + 2$ **[1 Mark]**

Substituting the value of y' in the given differential equation,

we get L.H.S.$= y' - 2x - 2 = 2x + 2 - 2x - 2 = 0 =$ R.H.S.

Hence, the given function is the solution of the corresponding differential equation. **[1 Mark]**

3. $y = \cos x + C : y' + \sin x = 0$ **[2 Marks]**

Solution:

Given equation is $y = \cos x + C$

Differentiating both sides of this equation with respect to x, we get:

$$y' = \frac{d}{dx}(\cos x + C)$$

$\Rightarrow y' = -\sin x$ **[1 Mark]**

Substituting the value of y' in the given differential equation, we get

$\text{L. H. S} = y' + \sin x = -\sin x + \sin x = 0 =$ R.H.S

Hence, the given function is the solution of the corresponding differential equation. **[1 Mark]**

4. $y = \sqrt{1+x^2}$: $y' = \frac{xy}{1+x^2}$ **[2 Marks]**

Solution:

Given equation is $y = \sqrt{1+x^2}$

Differentiating both sides of the equation with respect to x, we get:

$$y' = \frac{d}{dx}\left(\sqrt{1+x^2}\right)$$

$$\Rightarrow y' = \frac{1}{2\sqrt{1+x^2}} \cdot \frac{d}{dx}(1+x^2)$$

$$\Rightarrow y' = \frac{2x}{2\sqrt{1+x^2}}$$

$$\Rightarrow y' = \frac{x}{\sqrt{1+x^2}}$$

$$\Rightarrow y' = \frac{x}{1+x^2} \times \sqrt{1+x^2}$$

$$\Rightarrow y' = \frac{x}{1+x^2} \cdot y$$

$$\Rightarrow y' = \frac{xy}{1+x^2}$$

∴ L.H.S = R.H.S

Hence, the given function is the solution of the corresponding differential equation. **[2 Marks]**

5. $y = Ax : xy' = y (x \neq 0)$ **[1 Mark]**

Solution:

Given equation is $y = Ax$

Differentiating both sides with respect to x, we get:

$$y' = \frac{d}{dx}(Ax)$$

$$\Rightarrow y' = A$$

Substituting the value of y' in the given differential equation, we get:

L.H.S.=$xy' = x \cdot A = Ax = y$ =R.H.S

Hence, the given function is the solution of the corresponding differential equation. **[1 Mark]**

6. $y = x\sin x : xy' = y + x\sqrt{x^2 - y^2}(x \neq 0$ and $x > y$ or $x < -y)$ **[2 Marks]**

Solution:

Given equation is $y = x\sin x$

Differentiating both sides of this equation with respect to x, we get:

$$y' = \frac{d}{dx}(x\sin x)$$

$$\Rightarrow y' = \sin x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\sin x)$$

$$\Rightarrow y' = \sin x + x\cos x$$ **[1 Mark]**

Substituting the value of y' in the given differential equation, we get:

L.H.S = $xy' = x(\sin x + x\cos x)$

$= x\sin x + x^2\cos x$

$= y + x^2 \cdot \sqrt{1 - \sin^2 x}$

$= y + x^2\sqrt{1 - \left(\frac{y}{x}\right)^2}$

$= y + x\sqrt{y^2 - x^2}$

= R.H.S

Hence, the given function is the solution of the corresponding differential equation. **[1 Mark]**

7. $xy = \log y + C : y' = \frac{y^2}{1 - xy}(xy \neq 1)$ **[2 Marks]**

Solution:

Given equation is $xy = \log y + C$

Differentiating both sides of this equation with respect to x, we get:

$$\frac{d}{dx}(xy) = \frac{d}{dx}(\log y)$$

$$\Rightarrow y \cdot \frac{d}{dx}(x) + x \cdot \frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx}$$

$\Rightarrow y + xy' = \frac{1}{y}y'$ **[1 Mark]**

$$\Rightarrow y^2 + xyy' = y'$$

$$\Rightarrow (xy - 1)y' = -y^2$$

$$\Rightarrow y' = \frac{y^2}{1 - xy}$$

∴ L.H.S = R.H.S

Hence, the given function is the solution of the corresponding differential equation. **[1 Mark]**

8. $y = \cos y = x : (y \sin x + \cos y + x)y' = y$ **[2 Marks]**

Solution:

Given equation is $y - \cos y = x$

Differentiating both sides of the equation with respect to x, we get:

$$\frac{dy}{dx} - \frac{d}{dx}(\cos y) = \frac{d}{dx}(x)$$

$$\Rightarrow y' + \sin y \cdot y' = 1$$

$$\Rightarrow y'(1 + \sin y) = 1$$

$\Rightarrow y' = \frac{1}{1+\sin y}$ **[1 Mark]**

Substituting the value of y' in the L.H.S of the given differential equation, we get

$$= (y\sin y + \cos y + x)y'$$

$= (y\sin y + \cos y + y - \cos y) \times \frac{1}{1 + \sin y}$

$= y(1 + \sin y) \cdot \frac{1}{1 + \sin y}$

$= y$

$=$ R.H.S

Hence, the given function is the solution of the corresponding differential equation. **[1 Mark]**

9. $x + y = \tan^{-1} y : y^2 y' + y^2 + 1 = 0$
[2 Marks]

Solution:

Given equation is $x + y = \tan^{-1} y$

Differentiating both sides of this equation with respect to x, we get:

$\frac{d}{dx}(x + y) = \frac{d}{dx}(\tan^{-1} y)$

$\Rightarrow 1 + y' = \left[\frac{1}{1 + y^2}\right] y'$

$\Rightarrow y' \left[\frac{1}{1 + y^2} - 1\right] = 1$

$\Rightarrow y' \left[\frac{1 - (1 + y^2)}{1 + y^2}\right] = 1$

$\Rightarrow y' \left[\frac{-y^2}{1 + y^2}\right] = 1$

$\Rightarrow y' = \frac{-(1+y^2)}{y^2}$ **[1 Mark]**

Substituting the value of y' in the given differential equation, we get:

L.H.S.$= y^2 y' + y^2 + 1 = y^2 \left[\frac{-(1+y^2)}{y^2}\right] + y^2 + 1$

$= -1 - y^2 + y^2 + 1$

$= 0$

=R.H.S.

Hence, the given function is the solution of the corresponding differential equation. **[1 Mark]**

10. $y = \sqrt{a^2 - x^2} x \in (-a, a) : x + y\frac{dy}{dx} = 0 (y \neq 0)$ **[2 Marks]**

Solution:

Given function is $y = \sqrt{a^2 - x^2}$

Differentiating both sides of this equation with respect to x, we get:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{a^2 - x^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{a^2 - x^2}} \cdot \frac{d}{dx}(a^2 - x^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{a^2 - x^2}}(-2x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$$ **[1 Mark]**

Substituting the value of $\frac{dy}{dx}$

L.H.S.=$x + y\frac{dy}{dx} = x + \sqrt{a^2 - x^2} \times \frac{-x}{\sqrt{x^2 - x^2}}$

$= x - x$

$= 0$

=R.H.S.

Hence, the given function is the solution of the corresponding differential equation. **[1 Mark]**

11. The numbers of arbitrary constants in the general solution of a differential equation of fourth order are: **[1 Mark]**

(A) 0

(B) 2

(C) 3

(D) 4

Solution:

We know that the number of constants in the general solution of a differential equation of order n is equal to its order. **[$\frac{1}{2}$ Mark]**

Therefore, the number of constants in the general equation of fourth order differential equation is four.

Hence, the correct answer is D. **[$\frac{1}{2}$ Mark]**

12. The numbers of arbitrary constants in the particular solution of a differential equation of third order are: **[1 Mark]**

(A) 3

(B) 2

(C) 1

(D) 0

Solution:

In a particular solution of a differential equation, there will be no arbitrary constants. **[$\frac{1}{2}$ Mark]**

Hence, the correct answer is D. **[$\frac{1}{2}$ Mark]**

Exercise 9.3

In each of the Exercises 1 to 5, form a differential equation representing the given family of curves by eliminating arbitrary constants a and b.

1. $\frac{x}{a}+\frac{y}{b}=1$ **[2 Marks]**

Solution:

Given curve is $\frac{x}{a}+\frac{y}{b}=1$

Differentiating both sides of the given equation with respect to x, we get

$$\frac{1}{a}+\frac{1}{b}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{1}{a}+\frac{1}{b}y'=0$$

[1 Mark

Again, differentiating both sides with respect to x, we get

$$0+\frac{1}{b}y''=0$$

$$\Rightarrow \frac{1}{b}y''=0$$

$$\Rightarrow y''=0$$

Hence, the required differential equation of the given curve is $y''=0$ **[1 Mark]**

2. $y^2=a(b^2-x^2)$ **[2 Marks]**

Solution:

Given curve is $y^2=a(b^2-x^2)$

Differentiating both sides with respect to x, we get:

$$2y\frac{dy}{dx}=a(-2x)$$

$$\Rightarrow 2yy'=-2ax$$

$$\Rightarrow yy'=-ax \dots(1)$$

Again, differentiating both sides with respect to x, we get:

$$y'\cdot y'+yy''=-a$$

$\Rightarrow (y')^2 + yy'' = -a \ldots (2)$ **[1 Mark]**

Dividing equation (2) by equation (1), we get:

$$\frac{(y')^2 + yy''}{yy'} = \frac{-a}{-ax}$$

$\Rightarrow xyy'' + x(y')^2 - yy'' = 0$

Hence, the required differential equation of the given curve is $xyy'' + x(y')^2 - yy'' = 0$ **[1 Mark]**

3. $y = ae^{3x} + be^{-2x}$ **[4 Marks]**

Solution:

Given curve is $y = ae^{3x} + be^{-2x} \ldots (1)$

Differentiating both sides with respect to x, we get:

$y' = 3ae^{3x} - 2be^{-2x} \ldots (2)$

Again, differentiating both sides with respect to x, we get:

$y'' = 9ae^{3x} + 4be^{-2x} \ldots (3)$

Multiplying equation (1) with (2) and then adding it to equation (2), we get

$(2ae^{3x} + 2be^{-2x}) + (3ae^{3x} - 2bc^{-2x}) = 2y + y'$

$\Rightarrow 5ae^{3x} = 2y + y'$

$\Rightarrow ae^{3x} = \frac{2y+y'}{5}$ **[2 Marks]**

Now, multiplying equation (1) with equation (3) and subtracting equation (2) from it, we get:

$(3ae^{3x} + 3be^{-2x}) - (3ae^{3x} - 2be^{-2x}) = 3y - y'$

$\Rightarrow 5be^{-2x} = 3y - y'$

$\Rightarrow be^{-2x} = \frac{3y-y'}{5}$ **[1 Mark]**

Substituting the values of ae^{3x} and be^{-2x} in equation (3), we get:

$$y'' = 9 \cdot \frac{(2y + y')}{5} + 4\frac{(3y - y')}{5}$$

$$\Rightarrow y'' = \frac{18y + 9y'}{5} + \frac{12y - 4y'}{5}$$

$$\Rightarrow y'' = \frac{30y + 5y'}{5}$$

$$\Rightarrow y'' = 6y + y'$$

$$\Rightarrow y'' - y' - 6y = 0$$

Hence, the required differential equation of the given curve is $y'' - y' - 6y = 0$. **[1 Mark]**

4. $y = e^{2x}(a + bx)$ **[4 Marks]**

Solution:

Given curve is $y = e^{2x}(a + bx) \ldots (1)$

Differentiating both sides with respect to x, we get:

$$y' = 2e^{2x}(a + bx) + e^{2x} \cdot b$$

$$\Rightarrow y' = e^{2x}(2a + 2bx + b) \ldots (2)$$ **[1 Mark]**

Multiplying equation (1) with equation (2) and then subtracting it from equation (2), we get:

$$y' - 2y = e^{2x}(2a + 2bx + b) - e^{2x}(2a + 2bx)$$

$$\Rightarrow y' - 2 = be^{2x} \ldots (3)$$

Differentiating both sides with respect to x, we get:

$$y''k - 2y' = 2be^{2x} \ldots (4)$$ **[1 Mark]**

Dividing equation (4) by equation (3), we get:

$$\frac{y''k - 2y'}{y' - 2y} = 2$$

$$\Rightarrow y'' - 2y' = 2y' - 4y$$

$$\Rightarrow y'' - 4y' + 4y = 0$$

Hence, the required differential equation of the given curve is $y'' - 4y' + 4y = 0$ **[2 Marks]**

5. $y - e^{x}(a\cos x + b\sin x)$ **[4 Marks]**

Solution:

Given curve is $y - e^{x}(a\cos x + b\sin x) \ldots (1)$

Differentiating both sides with respect to x, we get:

$y' = e^{x}(a\cos x + b\sin x) + e^{x}(-a\sin x + b\cos x)$

$\Rightarrow y' = e^{x}[(a+b)\cos x - (a-b)\sin x] \ldots (2)$

Again, differentiating with respect to x, we get:

$y'' = e^{x}[(a+b)\cos x - (a-b)\sin x] + e^{x}[-(a+b)\sin x - (a-b)\cos x]$

$y'' = e^{x}[2b\cos x - 2a\sin x]$

$y'' = 2e^{x}(b\cos x - a\sin x)$

$\Rightarrow \frac{y''}{2} = e^{x}(b\cos x - a\sin x) \ldots (3)$

Adding equations (1) and (3), we get:

$y + \frac{y''}{2} = e^{x}[(a+b)\cos x - (a-b)\sin x]$ **[3 Marks]**

$\Rightarrow y + \frac{y''}{2} = y'$

$\Rightarrow 2y + y'' = 2y'$

$\Rightarrow y'' - 2y' + 2y = 0$

Hence, the required differential equation of the given curve is $y'' - 2y' + 2y = 0$ **[1 Mark]**

6. Form the differential equation of the family of circles touching the y-axis at the origin. **[2 Marks]**

Solution:

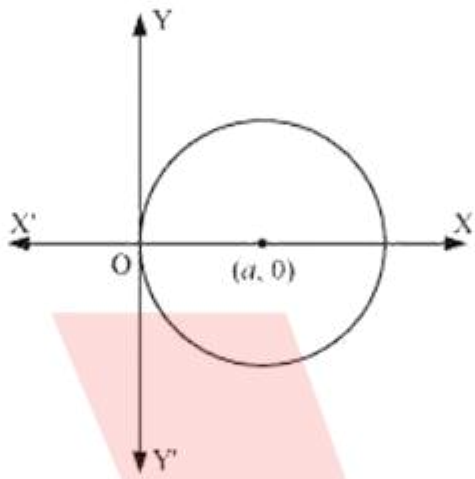
We know that the centre of the circle touching the y-axis at origin lies on the x-axis.

Let $(a, 0)$ be the centre of the circle.

Since it touches the y-axis at origin, its radius is a.

Now, the equation of the circle with centre $(a, 0)$ and radius a is $(x-a)^2 + y^2 = a^2$

$\Rightarrow x^2 + y^2 = 2ax \ldots (1)$



Now, differentiating equation (1) with respect to x, we get:

$2x + 2yy' = 2a$

$\Rightarrow x + yy' = a$ **[1 Mark]**

Now, on substituting the value of a in equation (1), we get:

$x^2 + y^2 = 2(x + yy')x$

$\Rightarrow x^2 + y^2 = 2x^2 + 2xyy'$

$\Rightarrow 2xyy' + x^2 = y^2$

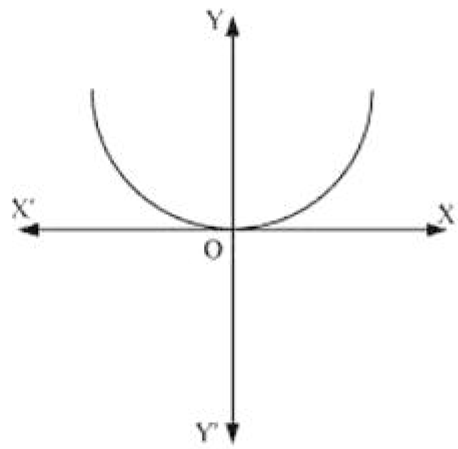
Hence, the required differential equation is $2xyy' + x^2 = y^2$ **[1 Mark]**

7. Form the differential equation of the family of parabolas having vertex at origin and axis along positive y-axis. **[2 Marks]**

Solution:

The equation of the parabola having the vertex at origin and the axis along the positive y-axis is

$x^2 = 4ay \ldots (1)$



Differentiating equation (1) with respect to x, we get:

$2x = 4ay' \dots (2)$ **[1 Mark]**

Dividing equation (2) by equation (1), we get:

$$\frac{2x}{x^2} = \frac{4ay'}{4ay}$$

$$\Rightarrow \frac{2}{x} = \frac{y'}{y}$$

$$\Rightarrow xy' = 2y$$

$$\Rightarrow xy' - 2y = 0$$

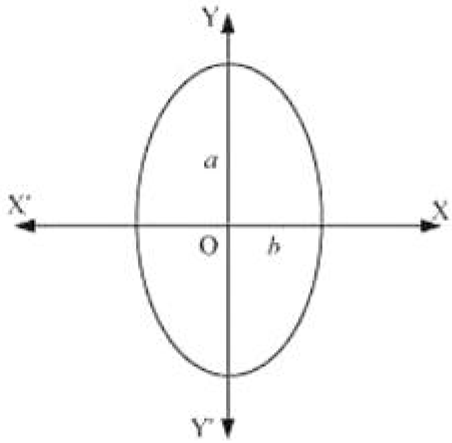
Hence, the required differential equation is $xy' - 2y = 0$. **[1 Mark]**

8. Form the differential equation of the family of ellipses having foci on y-axis and centre at origin.

[4 Marks]

Solution:

The equation of the family of ellipses having foci on the y-axis and the centre at origin is $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \dots (1)$



Differentiating equation (1) with respect to x, we get:

$\frac{2x}{b^2}+\frac{2yy'}{b^2}=0$ **[1 Mark]**

$\Rightarrow \frac{x}{b^2}+\frac{yy'}{a^2}=0 \ldots (2)$

Again, differentiating with respect to x, we get:

$\frac{1}{b^2}+\frac{y'\cdot y'+y\cdot y''}{a^2}=0$

$\Rightarrow \frac{1}{b^2}+\frac{1}{a^2}(y'^2+yy'')=0$

$\Rightarrow \frac{1}{b^2}=-\frac{1}{a^2}(y'^2+yy'')$ **[1 Mark]**

Substituting this value in equation (2), we get:

$x\left[-\frac{1}{a^2}((y')^2+yy'')\right]+\frac{yy'}{a^2}=0$ **[1 Mark]**

$\Rightarrow -x(y')^2-xyy''+yy'=0$

$\Rightarrow xyy''+x(y')^2-yy'=0$

Hence, the required differential equation is $xyy''+x(y')^2-yy'=0$ **[1 Mark]**

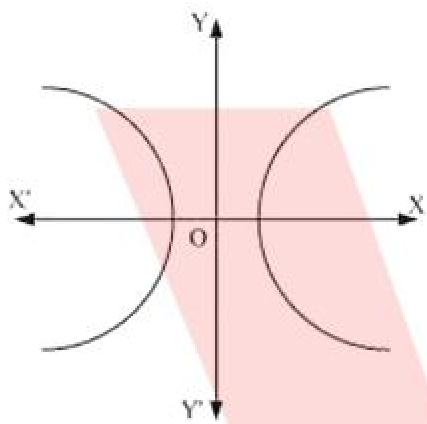
9. Form the differential equation of the family of hyperbolas having foci on x-axis and centre at origin.

[4 Marks]

Solution:

The equation of the family of hyperbolas with the centre at origin and foci along the x-axis is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \ldots (1)$$



Differentiating both sides of equation (1) with respect to x, we get:

$\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0$ **[1 Mark]**

$\Rightarrow \frac{x}{a^2} - \frac{yy'}{b^2} = 0 \ldots (2)$

Again, differentiating both sides with respect to x, we get:

$$\frac{1}{a^2} - \frac{y' \cdot y' + yy''}{b^2} = 0$$

$\Rightarrow \frac{1}{a^2} = \frac{1}{b^2}((y')^2 + yy'')$ **[1 Mark]**

Substituting the value of $\frac{1}{a^2}$ in equation (2)

$\frac{x}{b^2}((y')^2 + yy'') - \frac{yy'}{b^2} = 0$ **[1 Mark]**

$\Rightarrow x(y')^2 + xyy'' - yy' = 0$

$\Rightarrow xyy'' + x(y')^2 - yy' = 0$

Hence, the required differential equation is $xyy'' + x(y')^2 - yy' = 0$. **[1 Mark]**

10. Form the differential equation of the family of circles having centre on y-axis and radius 3 units.

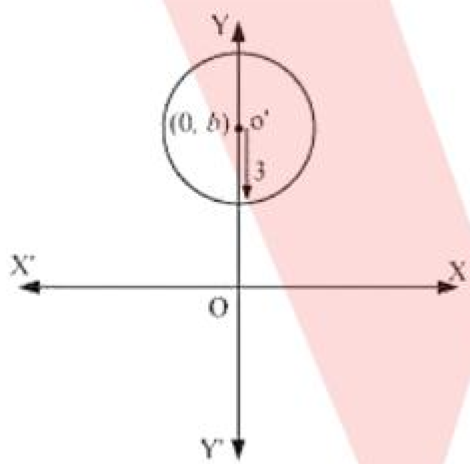
[2 Marks]

Solution:

Let the centre of the circle on y-axis be $(0, \mathrm{b})$.

The equation of the family of circles with centre at $(0, \mathrm{b})$ and radius 3 is $x^2 + (y-b)^2 = 3^2$

$\Rightarrow x^2 + (y-b)^2 = 9 \ldots (1)$



Differentiating equation (1) with respect to x, we get:

$2x + 2(y-b) \cdot y' = 0$

$\Rightarrow (y-b) \cdot y' = -x$

$\Rightarrow y - b = \frac{-x}{y'}$ **[1 Mark]**

Substituting the value of $(y - b)$ in equation (1), we get:

$$x^2 + \left(\frac{-x}{y'}\right)^2 = 9$$

$$\Rightarrow x^2\left[1 + \frac{1}{(y')^2}\right] = 9$$

$\Rightarrow x^2((y')^2 + 1) = 9(y')^2$

$\Rightarrow (x^2 - 9)(y')^2 + x^2 = 0$

Hence, the required differential equation is $(x^2 - 9)(y')^2 + x^2 = 0$ **[1 Mark]**

11. Which of the following differential equations has $y = c_1e^x + c_2e^{-x}$

as the general solution? **[2 Marks]**

(A) $\frac{d^2y}{dx^2} + y = 0$

(B) $\frac{d^2y}{dx^2} - y = 0$

(C) $\frac{d^2y}{dx^2} + 1 = 0$

(D) $\frac{d^2y}{dx^2} - 1 = 0$

Solution:

The given equation is $y = c_1e^x + c_2e^{-x}$ ....(i)

Differentiating with respect to x, we get:

$\frac{dy}{dx} = c_1e^x - c_2e^{-x}$ **[1 Mark]**

Again, differentiating with respect to x, we get:

$$\frac{d^2y}{dx^2} = c_1e^x + c_2e^{-x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = y$$

$$\Rightarrow \frac{d^2y}{dx^2} - y = 0$$

Hence, the required differential equation of the given equation of curve is $\frac{d^2y}{dx^2} - y = 0$ **[1 Mark]**

Hence, the correct answer is B.

12. Which of the following differential equation has $y = x$ as one of its particular solution? **[2 Marks]**

(A) $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = x$

(B) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = x$

(C) $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 0$

(D) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = 0$

The given equation of curve is $y = x$.

Differentiating with respect to x, we get:

$\frac{dy}{dx} = 1 \ldots (1)$

Again, differentiating with respect to x, we get:

$\frac{d^2y}{dx^2} = 0 \ldots (2)$ **[1 Mark]**

Now, on substituting the values of $y, \frac{d^2y}{dx^2}$, and $\frac{dy}{dx}$ from equation (1) and (2) in each of the given alternatives, we find that only the differential equation given in alternative C is correct.

$\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 0 - x^2 \cdot 1 + x \cdot x$

$= -x^2 + x^2$

$= 0$

Hence, the correct answer is C **[1 Mark]**

Exercise 9.4

For each of the differential equations in Exercises 1 to 10, find the general solution: **[2 Marks]**

1. $\frac{dy}{dx} = \frac{1-\cos x}{1+\cos x}$

Solution:

Given differential equation is $\frac{dy}{dx} = \frac{1-\cos x}{1+\cos x}$

$$\Rightarrow \frac{dy}{dx} = \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}} = \tan^2\frac{x}{2}$$

$$\Rightarrow \frac{dy}{dx} = \left(\sec^2\frac{x}{2} - 1\right)$$

Separating the variables, we get:

$dy = \left(\sec^2\frac{x}{2} - 1\right)dx$ **[1 Mark]**

Now, integrating both sides of this equation, we get

$$\int dy = \int\left(\sec^2\frac{x}{2} - 1\right)dx = \int \sec^2\frac{x}{2}dx - \int dx$$

$$\Rightarrow y = 2\tan\frac{x}{2} - x + C$$

Hence, the required general solution of the given differential equation is $y = 2\tan\frac{x}{2} - x + C$ **[1 Mark]**

2. $\frac{dy}{dx} = \sqrt{4-y^2}\,(-2 < y < 2)$ **[2 Marks]**

Solution:

Given differential equation is $\frac{dy}{dx} = \sqrt{4-y^2}$

Separating the variables, we get:

$\Rightarrow \frac{dy}{\sqrt{4-y^2}} = dx$ **[1 Mark]**

Now, integrating both sides of this equation, we get:

$$\int \frac{dy}{\sqrt{4-y^2}} = \int dx$$

$$\Rightarrow \sin^{-1}\frac{y}{2} = x + C$$

$\Rightarrow \frac{y}{2} = \sin(x + C)$

$\Rightarrow y = 2\sin(x + C)$

Hence, the required general solution of the given differential equation is $y = 2\sin(x + C)$ **[1 Mark]**

3. $\frac{dy}{dx} + y = 1 (y \neq 1)$ **[2 Marks]**

Solution:

Given differential equation is $\frac{dy}{dx} + y = 1$

$\Rightarrow dy + y\,dx = dx$

$\Rightarrow dy = (1 - y)dx$

Separating the variables, we get:

$\Rightarrow \frac{dy}{1-y} = dx$ **[1 Mark]**

Now, integrating both sides, we get:

$$\int \frac{dy}{1-y} = \int d\,x$$

$\Rightarrow \log(1 - y) = x + \log C$

$\Rightarrow -\log C - \log(1 - y) = x$

$\Rightarrow \log C(1 - y) = -x$

$\Rightarrow C(1 - y) = e^{-x}$

$\Rightarrow 1 - y = \frac{1}{C}e^{-x}$

$\Rightarrow y = 1 - \frac{1}{C}e^{-x}$

$\Rightarrow y = 1 + Ae^{-x} \left(\text{where } A = -\frac{1}{C}\right)$

Hence, the required general solution of the given differential equation is $y = 1 + Ae^{-x}$ **[1 Mark]**

4. $\sec^2 x \tan y\, dx + \sec^2 y\, \tan x\, dy = 0$ **[4 Marks]**

Solution:

Given differential equation is $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$

$$\Rightarrow \frac{\sec^2 x \tan y dx + \sec^2 y \tan x dy}{\tan x \tan y} = 0$$

$$\Rightarrow \frac{\sec^2 x}{\tan x} dx + \frac{\sec^2 y}{\tan y} dy = 0$$

$$\Rightarrow \frac{\sec^2 x}{\tan x} dx = -\frac{\sec^2 y}{\tan y} dy$$

Integrating both sides of this equation, we get:

$\int dx = -\int \frac{\sec^2 y}{\tan y} dy \ldots (1)$ **[1 Mark]**

Let $\tan x = t$

$$\therefore \frac{d}{dx}(\tan x) = \frac{dt}{dx}$$

$$\Rightarrow \sec^2 x = \frac{dt}{dx}$$

$$\Rightarrow \sec^2 x dx = dt$$

Now, $\int \frac{\sec^2 x}{\tan x} dx = \int \frac{1}{t} dt$

$= \log t$

$= \log(\tan x)$ **[1 Mark]**

Similarly, $\int \frac{\sec^2 x}{\tan x} dy = \log(\tan y)$

Substituting these values in equation (1), we get:

$$\log(\tan x) = -\log(\tan y) + \log \text{C}$$

$$\Rightarrow \log(\tan x) = \log\left(\frac{\text{C}}{\tan y}\right)$$

$$\Rightarrow \tan x = \frac{\text{C}}{\tan y}$$

$$\Rightarrow \tan x \tan y = \text{C}$$

Hence, the required general solution of the given differential equation is $\tan x \tan y = C$ **[2 Marks]**

5. $(e^x + e^{-x})dy - (e^x - e^{-x})dx = 0$ **[2 Marks]**

Solution:

Given differential equation is $(e^x + e^{-x})dy - (e^x - e^{-x})dx = 0$

$\Rightarrow (e^x + e^{-x})dy = (e^x - e^{-x})dx$

$\Rightarrow dy = \left[\frac{e^x - e^{-x}}{e^x + e^{-x}}\right]dx$

Integrating both sides of this equation, we get:

$$\int dy = \int \left[\frac{e^x - e^{-x}}{e^x + e^{-x}}\right]dx + C$$

$\Rightarrow y = \int \left[\frac{e^x - e^{-x}}{e^x + e^{-x}}\right]dx + C \ldots (1)$

Let $e^x + e^{-x}) = t$

Differentiating both sides with respect to x, we get:

$$\frac{d}{dx}(e^x + e^{-x}) = \frac{dt}{dx}$$

$\Rightarrow e^x - e^{-x} = \frac{dt}{dt}$

$\Rightarrow (e^x - e^{-x})dx = dt$ **[1 Mark]**

Substituting this value in equation (1), we get:

$$y = \int \frac{1}{t}dt + C$$

$\Rightarrow y = \log(t) + C$

$\Rightarrow y = \log(e^x + e^{-x}) + C$

Hence,

the required general solution of the given differential equation is $y = \log(e^x + e^{-x}) + C$**[1 Mark]**

6. $\frac{dy}{dx} = (1 + x^2)(1 + y^2)$ **[2 Marks]**

Solution:

Given differential equation is $\frac{dy}{dx} = (1 + x^2)(1 + y^2)$

$\Rightarrow \frac{dy}{1+y^2} = (1 + x^2)dx$ **[1 Mark]**

Integrating both sides of this equation, we get:

$$\int \frac{dy}{1 + y^2} = \int (1 + x^2)\, dx$$

$$\Rightarrow \tan^{-1} y = \int dx + \int x^2 dx$$

$$\Rightarrow \tan^{-1} y = x + \frac{x^3}{3} + C$$

Hence, the required general solution of the given differential equation is $\tan^{-1} y = x + \frac{x^3}{3} + C$ **[1 Mark]**

7. $y\log y dx - xdy = 0$ **[2 Marks]**

Solution:

Given differential equation is $y\log y dx - xdy = 0$

$\Rightarrow y\log y dx = xdy$

$$\Rightarrow \frac{dy}{y\log y} = \frac{dx}{x}$$

Integrating both sides, we get:

$$\int \frac{dy}{y\log y} = \int \frac{dx}{x} \dots (1)$$

Let $\log y = t$

$$\therefore \frac{d}{dy}(\log y) = \frac{dt}{dy}$$

$\Rightarrow \frac{1}{y} = \frac{dt}{dy}$

$\Rightarrow \frac{1}{y} dy = dt$ **[1 Mark]**

Substituting this value in equation (1), we get:

$\int \frac{dt}{t} = \int \frac{dx}{x}$

$\Rightarrow \log t = \log x + \log \mathrm{C}$

$\Rightarrow \log(\log y) = \log Cx$

$\Rightarrow \log y = \mathrm{C}x$

$\Rightarrow y = e^{cx}$

Hence, the required general solution of the given differential equation is $y = e^{cx}$ **[1 Mark]**

8. $x^5 \frac{dy}{dx} = -y^5$ **[2 Marks]**

Solution:

Given differential equation is $x^5 \frac{dy}{dx} = -y^5$

$\Rightarrow \frac{dy}{y^5} = -\frac{dx}{x^5}$

$\Rightarrow \frac{dx}{x^5} + \frac{dy}{y^5} = 0$

Integrating both sides, we get:

$\int \frac{dx}{x^5} + \int \frac{dy}{y^5} = k$ **[1 Mark]**

$\Rightarrow \int x^{-5} \, dx + \int y^{-5} \, dy = k$

$\Rightarrow \frac{x^{-4}}{-4} + \frac{y^{-4}}{-4} = k$

$\Rightarrow x^{-4} + y^{-4} = -4k$

$\Rightarrow x^{-4} + y^{-4} = C \; (C = -4k)$

Hence, the required general solution of the given differential equation is $x^{-4} + y^{-4} = C$ **[1 Mark]**

9. $\frac{dy}{dx} = \sin^{-1} x$

[4 Marks]

Solution:

Given differential equation is $\frac{dy}{dx} = \sin^{-1} x$

$\Rightarrow dy = \sin^{-1} x \, dx$

Integrating both sides, we get:

$$\int dy = \int \sin^{-1} x dx$$

$$\Rightarrow y = \int (\sin^{-1}x \cdot 1) \, dx$$

$$\Rightarrow y = \sin^{-1}x \cdot \int (1) \, dx - \int \left[\left(\frac{d}{dx} (\sin^{-1}x) \cdot \int (1) \, dx \right) \right] dx$$

$$\Rightarrow y = \sin^{-1}x \cdot x - \int \left(\frac{1}{\sqrt{1 - x^2}} \cdot x \right) dx$$

$\Rightarrow y = x\sin^{-1}x + \int \frac{-x}{\sqrt{1-x^2}} dx \dots (1)$ **[1 Mark]**

Let $1 - x^2 = t$

$$\Rightarrow \frac{d}{dx}(1 - x^2) = \frac{dt}{dx}$$

$$\Rightarrow -2x = \frac{dt}{dx}$$

$$\Rightarrow xdx = -\frac{1}{2} dt$$

Substituting this value in equation (1), we get:

$y = x\sin^{-1}x + \int \frac{1}{2\sqrt{t}} dt$ **[1 Mark]**

$$\Rightarrow y = x\sin^{-1}x + \frac{1}{2} \cdot \int (t)^{-\frac{1}{2}} dt$$

$\Rightarrow y = x\sin^{-1}x + \frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + \text{C}$

$\Rightarrow y = x\sin^{-1}x + \sqrt{t} + \text{C}$

$\Rightarrow y = x\sin^{-1}x + \sqrt{1-x^2} + \text{C}$ **[2 Marks]**

Hence, the required general solution of the given differential equation is $y = x\sin^{-1}x + \sqrt{1-x^2} + \text{C}$

10. $e^x\tan y dx + (1-e^x)\sec^2 y dy = 0$ **[4 Marks]**

Solution:

Given differential equation is $e^x\tan y dx + (1-e^x)\sec^2 y dy = 0$

$(1-e^x)\sec^2 y dy = -e^x\tan y dx$

Separating the variables, we get:

$\frac{\sec^2 y}{\tan y}dy = \frac{-e^x}{1-e^x}dx$

Integrating both sides, we get:

$\int\frac{\sec^2 y}{\tan y}dy = \int\frac{-e^x}{1-e^x}dx \ldots(1)$ **[1 Mark]**

Let $\tan y = u$

$\Rightarrow \frac{d}{dy}(\tan y) = \frac{du}{dy}$

$\Rightarrow \sec^2 y = \frac{du}{dy}$

$\Rightarrow \sec^2 y dy = du$

$\therefore \int\frac{\sec^2 y}{\tan y}dy = \int\frac{du}{u} = \log u = \log(\tan y)$ **[1 Mark]**

Now, Let $1-e^x = t$.

$\therefore \frac{d}{dx}(1-e^x) = \frac{dt}{dx}$

$\Rightarrow -e^x = \frac{dt}{dx}$

$\Rightarrow -e^x dx = dt$

$\Rightarrow \int \frac{-e^x}{1-e^x} dx = \int \frac{dt}{t} = \log t = \log(1-e^x)$ **[1 Mark]**

Substituting the values of $\int \frac{\sec^2 y}{\tan y} dy$ and $\int \frac{-e^x}{1-e^x} dx$

$\Rightarrow \log(\tan y) = \log(1-e^x) + \log C$

$\Rightarrow \log(\tan y) = \log[C(1-e^x)]$

$\Rightarrow \tan y = C(1-e^x)$

Hence, the required general solution of the given differential equation is $\tan y = C(1-e^x)$**[1 Mark]**

For each of the differential equations in Exercises 11 to 14, find a particular solution satisfying the given condition:

11. $(x^3 + x^2 + x + 1)\frac{dy}{dx} = 2x^2 + x; y = 1$ when $x = 0$

[6 Marks]

Solution:

Given differential equation is $(x^3 + x^2 + x + 1)\frac{dy}{dx} = 2x^2 + x$

$$\Rightarrow \frac{dy}{dx} = \frac{2x^2 + x}{(x^3 + x^2 + x + 1)}$$

$$\Rightarrow dy = \frac{2x^2 + x}{(x+1)(x^2+1)} dx$$

Integrating both sides, we get:

$$\int dy = \int \frac{2x^2 + x}{(x+1)(x^2+1)} dx \dots (1)$$

Let $\frac{2x^2+x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \dots (2)$

$\Rightarrow \frac{2x^2+x}{(x+1)(x^2+1)} = \frac{Ax^2+A+(Bx+C)(x+1)}{(x+1)(x^2+1)}$ **[1 Mark]**

$\Rightarrow 2x^2 + x = Ax^2 + A + Bx^2 + Bx + Cx + C$

$\Rightarrow 2x^2 + x = (A+B)x^2 + (B+C)x + (A+C)$

Comparing the coefficients of x^2 and x, we get:

$A + B = 2$

$B + C = 1$

$A + C = 0$

Solving these equations, we get:

$A = \frac{1}{2}, B = \frac{3}{2}$ and $C = \frac{-1}{2}$ **[1 Mark]**

Substituting the values of A, B, and C in equation (2), we get:

$$\frac{2x^2 + x}{(x+1)(x^2+1)} = \frac{1}{2} \cdot \frac{1}{(x+1)} + \frac{1}{2}\frac{(3x-1)}{(x^2+1)}$$

Therefore, equation (1) becomes:

$\int dy = \frac{1}{2}\int \frac{1}{x+1} dx + \frac{1}{2}\int \frac{3x-1}{x^2+1} dx$ **[1 Mark]**

$$\Rightarrow y = \frac{1}{2}\log(x+1) + \frac{3}{2}\int \frac{x}{x^2+1} dx - \frac{1}{2}\int \frac{1}{x^2+1} dx$$

$$\Rightarrow y = \frac{1}{2}\log(x+1) + \frac{3}{4} \cdot \int \frac{2x}{x^2+1} dx - \frac{1}{2}\tan^{-1}x + C$$

$$\Rightarrow y = \frac{1}{2}\log(x+1) + \frac{3}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1}x + C$$

$$\Rightarrow y = \frac{1}{4}[2\log(x+1) + 3\log(x^2+1)] - \frac{1}{2}\tan^{-1}x + C$$

$\Rightarrow y = \frac{1}{4}[(x+1)^2(x^2+1)^3] - \frac{1}{2}\tan^{-1}x + C \ldots (3)$ **[1 Mark]**

Now, $y = 1$ when $x = 0$

$$\Rightarrow 1 = \frac{1}{4}\log(1) - \frac{1}{2}\tan^{-1}0 + C$$

$$\Rightarrow 1 = \frac{1}{4} \times 0 - \frac{1}{2} \times 0 + C$$

$\Rightarrow C = 1$ **[1 Mark]**

Substituting $C = 1$ in equation (3), we get:

$$y = \frac{1}{4}[\log(x+1)^2(x^2+1)^3] - \frac{1}{2}\tan^{-1}x + 1$$

Hence, the required general solution of the given differential equation is $y = \frac{1}{4}[\log(x+1)^2(x^2+1)^3] - \frac{1}{2}\tan^{-1}x + 1$

[1 Mark]

12. $x(x^2-1)\frac{dy}{dx} = 1; y = 0$ when $x = 2$ **[6 Marks]**

Solution:

Given differential equation is $x(x^2-1)\frac{dy}{dx} = 1$

$$\Rightarrow dy = \frac{dx}{x(x^2-1)}$$

$$\Rightarrow dy = \frac{1}{x(x-1)(x+1)}dx$$

Integrating both sides, we get:

$\int dy = \int \frac{1}{x(x-1)(x+1)}dx \ldots (1)$ **[1 Mark]**

Let $\frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \ldots (2)$

$\Rightarrow \frac{1}{x(x-1)(x+1)} = \frac{A(x-1)(x+1)+Bx(x+1)+Cx(x-1)}{x(x-1)(x+1)}$ **[1 Mark]**

$$= \frac{(A+B+C)x^2 + (B-C)x - A}{x(x-1)(x+1)}$$

Comparing the coefficients of x^2, x, and constant, we get:

$A = -1$

$B - C = 0$

$A + B + C = 0$

Solving these equations, we get

$B = \frac{1}{2}$ and $C = \frac{1}{2}$ **[1 Mark]**

Substituting the values of A, B, and C in equation (2), we get:

$$\frac{1}{x(x-1)(x+1)}=\frac{-1}{x}+\frac{1}{2(x-1)}+\frac{1}{2(x+1)}$$

Therefore, equation (1) becomes:

$$\int dy=-\int\frac{1}{x}dx+\frac{1}{2}\int\frac{1}{x-1}dx+\frac{1}{2}\int\frac{1}{x+1}dx$$

$\Rightarrow y=-\log x+\frac{1}{2}\log(x-1)+\frac{1}{2}\log(x+1)+\log k$ **[1 Mark]**

$$\Rightarrow y=\frac{1}{2}\log\left[\frac{k^2(x-1)(x+1)}{x^2}\right] \dots (3)$$

Now, $y=0$ when $x=2$

$$\Rightarrow 0=\frac{1}{2}\log\left[\frac{k^2(2-1)(2+1)}{4}\right]$$

$$\Rightarrow \log\left(\frac{3k^2}{4}\right)=0$$

$$\Rightarrow \frac{3k^2}{4}=1$$

$$\Rightarrow 3k^2=4$$

$\Rightarrow k^2=\frac{4}{3}$ **[1 Mark]**

Substituting the value of k^2 in equation (3), we get:

$$y=\frac{1}{2}\log\left[\frac{4(x-1)(x+1)}{3x^2}\right]$$

$$\Rightarrow y=\frac{1}{2}\log\left[\frac{4(x^2-1)}{3x^2}\right]$$

Hence, the required general solution of the given differential equation is $y=\frac{1}{2}\log\left[\frac{4(x^2-1)}{3x^2}\right]$ **[1 Mark]**

13. $\cos\left(\frac{dy}{dx}\right)=a(a\in R); y=1$ when $x=0$ **[2 Marks]**

Solution:

Given differential equation is $\cos\left(\frac{dy}{dx}\right)=a$

$\Rightarrow \dfrac{dy}{dx} = \cos^{-1}a$

$\Rightarrow dy = \cos^{-1}a dx$

Integrating both sides, we get:

$\int dy = \cos^{-1}a \int dx$

$\Rightarrow y = \cos^{-1}a \cdot x + C$

$\Rightarrow y = x\cos^{-1}a + C \dots (1)$

Now, $y = 1$ when $x = 0$

$\Rightarrow 1 = 0\cos^{-1}a + C$

$\Rightarrow C = 1$ **[1 Mark]**

Substituting $C = 1$ in equation (1), we get:

$y = x\cos^{-1}a + 1$

$\Rightarrow \dfrac{y-1}{x} = \cos^{-1}a$

$\Rightarrow \cos\left(\dfrac{y-1}{x}\right) = a$

Hence, the required general solution of the given differential equation is $\cos\left(\frac{y-1}{x}\right) = a$ **[1 Mark]**

14. $\frac{dy}{dx} = y\tan x; y = 1$ when $x = 0$ **[2 Marks]**

Solution:

Given differential equation is $\frac{dy}{dx} = y\tan x$

$\Rightarrow \dfrac{dy}{y} = \tan x dx$

Integrating both sides, we get:

$\int \dfrac{dy}{y} = -\int \tan x \, dx$

$\Rightarrow \log y = \log(\sec x) + \log C$

$\Rightarrow \log y = \log(C \sec x)$

$\Rightarrow y = C\sec x \ldots (1)$ **[1 Mark]**

Now, $y = 1$ when $x = 0$

$\Rightarrow 1 = C \times \sec 0$

$\Rightarrow 1 = C \times 1$

$\Rightarrow C = 1$

Substituting $C = 1$ in equation (1), we get:

$y = \sec x$

$\Rightarrow y - \sec x = 0$

Hence, the required general solution of the given differential equation is $y - \sec x = 0$ **[1 Mark]**

15. Find the equation of a curve passing through the point (0, 0) and whose differential equation is. $y' = e^x \sin x$ **[4 Marks]**

Solution:

Given that the differential equation of the curve is $y' = e^x \sin x$

$$\Rightarrow \frac{dy}{dx} = e^x \sin x$$

$$\Rightarrow dy = e^x \sin x$$

Integrating both sides, we get:

$\int d\, y = \int e^x \sin x dx \ldots (1)$ **[1 Mark]**

$$I = \int e^x \sin x dx$$

$$\Rightarrow I = \sin x \int e^x\, dx - \int \left(\frac{d}{dx}(\sin x) \cdot \int e^x\, dx\right) dx$$

$$\Rightarrow I = \sin x \cdot e^x - \int \cos x \cdot e^x dx$$

$$\Rightarrow I = \sin x \cdot e^x - \left[\cos x \cdot \int e^x dx - \int \left(\frac{d}{dx}(\cos x) \cdot \int e^x\, dx\right) dx\right]$$

$\Rightarrow I = \sin x \cdot e^x - \left[\cos x \cdot e^x - \int(-\sin x) \cdot e^x dx\right]$

$\Rightarrow I = e^x \sin x - e^x \cos x - I$

$\Rightarrow 2I = e^x(\sin x - \cos x)$

$\Rightarrow I = \frac{e^x(\sin x - \cos x)}{2}$ **[1 Mark]**

Substituting this value in equation (1), we get:

$$y = \frac{e^x(\sin x - \cos x)}{2} + C \ldots (2)$$

Now the curve passes through point $(0, 0)$

$$\therefore 0 = \frac{e^0(\sin 0 - \cos 0)}{2} + C$$

$$\Rightarrow 0 = \frac{1(0-1)}{2} + C$$

$\Rightarrow C = \frac{1}{2}$ **[1 Mark]**

Substituting $C = \frac{1}{2}$ in equation (2), we get:

$$y = \frac{e^x(\sin x - \cos x)}{2} + \frac{1}{2}$$

$$\Rightarrow 2y = e^x(\sin x - \cos x) + 1$$

$$\Rightarrow 2y - 1 = e^x(\sin x - \cos x)$$

Hence, the required equation of the given curve is $2y - 1 = e^x(\sin x - \cos x)$ **[1 Mark]**

16. For the differential equation find the solution curve passing through the point (1, −1). **[4 Marks]**

Solution:

Given differential equation of the curve is $xy\frac{dy}{dx} = (x+2)(y+2)$

$$\Rightarrow \left(\frac{y}{y+2}\right)dy = \left(\frac{x+2}{x}\right)dx$$

$$\Rightarrow \left(1 - \frac{2}{y+2}\right)dy = \left(1 + \frac{2}{x}\right)$$

Integrating both sides, we get:

$$\int\left(1-\frac{2}{y+2}\right)dy=\int\left(1+\frac{2}{x}\right)dx$$

$$\Rightarrow \int dy-2\int\frac{1}{y+2}dy=\int dx+2\int\frac{1}{x}dx$$

$$\Rightarrow y-2\log(y+2)=x+2\log x+\mathrm{C}$$

$$\Rightarrow y-x-\mathrm{C}=\log x^2+\log(y+2)^2$$

$$\Rightarrow y-x-\mathrm{C}=\log[x^2(y+2)^2]\ \ldots(1)$$

[2 Marks]

Now, the curve passed through point $(1,-1)$

$$\Rightarrow -1-1-C=\log[(1)^2(-1+2)^2]$$

$$\Rightarrow -2-C=\log 1=0$$

$$\Rightarrow C=-2$$

[1 Mark]

Substituting $C=-2$ in equation (1), we get:

$$y-x+2=\log[x^2(y+2)^2]$$

Hence, the required solution of the given curve is $y-x+2=\log[x^2(y+2)^2]$ **[1 Mark]**

17. Find the equation of a curve passing through the point $(0,-2)$ given that at any point (x,y) on the curve, the product of the slope of its tangent and y-coordinate of the point is equal to the x-coordinate of the point. **[4 Marks]**

Solution:

Let x and y be the x-coordinate and y-coordinate of the curve respectively.

We know that the slope of a tangent to the curve in the coordinate axis is given by the relation,

$$\frac{dy}{dx}$$

According to the given information, we get:

$$y\cdot\frac{dy}{dx}=x$$

$\Rightarrow ydy = xdx$

Integrating both sides, we get:

$$\int ydy = \int xdx$$

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C$$

$\Rightarrow y^2 - x^2 = 2C \dots (1)$ **[2 Marks]**

Now, the curve passes through point $(0, -2)$.

$\therefore (-2)^2 - 0^2 = 2C$

$\Rightarrow 2C = 4$ **[1 Mark]**

Substituting $2C = 4$ in equation (1), we get:

$y^2 - x^2 = 4$

Hence, the required equation of the curve is $y^2 - x^2 = 4$ **[1 Mark]**

18. At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point $(-4, -3)$. Find the equation of the curve given that it passes through $(-2, 1)$. **[4 Marks]**

Solution:

Given that (x, y) is the point of contact of the curve and its tangent.

Let m_1 be the slope of the line segment joining the given points and m_2 be the slope of the tangent

We know, slope (m_1) of the line segment joining (x, y) and $(-4, -3)$ is $\frac{x+3}{x+4}$

And slope (m_2) of the tangent = $\frac{dy}{dx}$

According to the given information:

$m_2 = 2m_1$

$$\Rightarrow \frac{dy}{dx} = \frac{2(y+3)}{x+4}$$

$\Rightarrow \frac{dy}{y+3} = \frac{2dx}{x+4}$

Integrating both sides, we get:

$\int \frac{dy}{y+3} = 2\int \frac{dx}{x+4}$

$\Rightarrow \log(y+3) = 2\log(x+4) + \log C$

$\Rightarrow \log(y+3) \log C(x+4)^2$

$\Rightarrow y+3 = C(x+4)^2 \ldots (1)$ **[2 Marks]**

This is the general equation of the curve.

It is given that it passes through point $(-2, 1)$.

$\Rightarrow 1 + 3 = C(-2+4)^2$

$\Rightarrow 4 = 4C$

$\Rightarrow C = 1$ **[1 Mark]**

Substituting $C = 1$ in equation (1), we get:

$y + 3 = (x+4)^2$

Hence, the required equation of the curve is $y + 3 = (x+4)^2$ **[1 Mark]**

19. The volume of spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of balloon after t seconds. **[4 Marks]**

Solution:

Let the rate of change of the volume of the balloon be k (where k is a constant).

$\Rightarrow \frac{dv}{dt} = k$

$\Rightarrow \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = k \left[\text{Volume of sphere} = \frac{4}{3}\pi r^3\right]$

$\Rightarrow \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt} = k$

$\Rightarrow 4\pi r^2 dr = kdt$ **[1 Mark]**

Integrating both sides, we get:

$$4\pi \int r^2\, dr = k \int dt$$

$$\Rightarrow 4\pi \cdot \frac{r^3}{3} = kt + \text{C}$$

$$\Rightarrow 4\pi r^3 = 3(kt + \text{C}) \ldots (1)$$

Now, at $t = 0, r = 3$:

$$\Rightarrow 4\pi \times 3^3 = 3(k \times 0 + C)$$

$$\Rightarrow 108\pi = 3C$$

$\Rightarrow C = 36\pi$ **[1 Mark]**

At $t = 3, r = 6$:

$$\Rightarrow 4\pi \times 6^3 = 3(k \times 3 + C)$$

$$\Rightarrow 864\pi = 3(3k + 36\pi)$$

$$\Rightarrow 3k =- 288\pi - 36\pi = 252\pi$$

$\Rightarrow k = 84\pi$ **[1 Mark]**

Substituting the values of k and C in equation (1), we get:

$$4\pi r^3 = 3[84\pi t + 36\pi]$$

$$\Rightarrow 4\pi r^3 = 4\pi(63t + 27)$$

$$\Rightarrow r^3 = 63t + 27$$

$$\Rightarrow r = (63t + 27)^{\frac{1}{3}}$$

Thus, the radius of the balloon after t seconds is $(63t + 27)^{\frac{1}{3}}$ **[1 Mark]**

20. In a bank, principal increases continuously at the rate of $r\%$ per year. Find the value of r if Rs 100 doubles itself in 10 years $(\log_e 2 = 0.6931)$. **[4 Marks]**

Solution:

Let p and t represent the principal and time respectively.

It is given that the principal increases continuously at the rate of $r\%$ per year.

$$\Rightarrow \frac{dp}{dt} = \left(\frac{r}{100}\right)p$$

$$\Rightarrow \frac{dp}{p} = \left(\frac{r}{100}\right)dt$$

Integrating both sides, we get:

$$\int \frac{dp}{p} = \frac{r}{100}\int dt$$

$$\Rightarrow \log p = \frac{rt}{100} + k$$

$\Rightarrow p = e^{\frac{rt}{100}+k} \ldots (1)$ **[1 Mark]**

It is given that when $t = 0, p = 100$.

$\Rightarrow 100 = e^k \ldots (2)$ **[1 Mark]**

Now, if $t = 10$, then $p = 2 \times 100 = 200$.

Therefore, equation (1) becomes:

$$200 = e^{\frac{r}{10}+k}$$

$$\Rightarrow 200 = e^{\frac{r}{10}} \cdot e^k$$

$$\Rightarrow 200 = e^{\frac{r}{10}} \cdot 100 \text{ (From (2))}$$

$$\Rightarrow e^{\frac{r}{10}} = 2$$

$$\Rightarrow \frac{r}{10} = \log_e 2$$

$$\Rightarrow \frac{r}{10} = 0.6931$$

$$\Rightarrow r = 0.6931$$

Hence, the value of r is 6.93% **[2 Marks]**

21. In a bank, principal increases continuously at the rate of 5% per year. An amount of Rs

1000 is deposited with this bank, how much will it worth after 10 years ($e^{0.5} = 1.648$) **[4 Marks]**

Solution:

Let p and t be the principal and time respectively.

It is given that the principal increases continuously at the rate of 5% per year.

$$\Rightarrow \frac{dp}{dt} = \left(\frac{5}{100}\right)p$$

$$\Rightarrow \frac{dp}{dt} = \frac{p}{20}$$

$$\Rightarrow \frac{dp}{p} = \frac{dt}{20}$$

[1 Mark]

Integrating both sides, we get:

$$\int \frac{dp}{p} = \frac{1}{20}\int dt$$

$$\Rightarrow \log p = \frac{t}{20} + C$$

$$\Rightarrow p = e^{\frac{t}{20}+C} \ldots(1)$$

[1 Mark]

Now, when $t = 0$, $p = 1000$.

$$\Rightarrow 1000 = e^{C} \ldots(2)$$

[1 Mark]

At $t = 10$, equation (1) becomes:

$$p = e^{\frac{1}{2}+C}$$

$$\Rightarrow p = e^{0.5} \times e^{C}$$

$$\Rightarrow p = 1.648 \times 1000$$

$$\Rightarrow p = 1648$$

Hence, after 10 years the amount will worth Rs 1648. **[1 Mark]**

22. In a culture, the bacteria count is 1,00,000. The number is increased by 10% in 2 hours. In how many hours will the count reach 2,00,000, if the rate of growth of bacteria is

proportional to the number present?
[6 Marks]

Solution:

Let y be the number of bacteria at any instant t.

It is given that the rate of growth of the bacteria is proportional to the number present.

$\therefore \frac{dy}{dt} \propto y$

$\Rightarrow \frac{dy}{dt} = ky$ (where k is a constant)

$\Rightarrow \frac{dy}{y} = kdt$ **[1 Mark]**

Integrating both sides, we get:

$\int \frac{dy}{y} = k \int dt$

$\Rightarrow \log y = kt + C$ (1) **[1 Mark]**

Let y_0 be the number of bacteria at $t = 0$.

$\Rightarrow \log y_0 = C$

Substituting the value of C in equation (1), we get:

$\log y = kt + \log y_0$

$\Rightarrow \log y - \log y_0 = kt$

$\Rightarrow \log\left(\frac{y}{y_0}\right) = kt$

$\Rightarrow kt = \log\left(\frac{y}{y_0}\right) \dots (2)$ **[1 Mark]**

Also, it is given that the number of bacteria increases by 10% in 2 hours.

$\Rightarrow y = \frac{110}{100} y_0$

$\Rightarrow \frac{y}{y_0} = \frac{11}{10} \dots (3)$ **[1 Mark]**

Substituting this value in equation (2), we get:

$k \cdot 2 = \log\left(\frac{11}{10}\right)$

$\Rightarrow k = \frac{1}{2}\log\left(\frac{11}{10}\right)$

Therefore, equation (2) becomes:

$\frac{1}{2}\log\left(\frac{11}{10}\right) \cdot t = \log\left(\frac{y}{y_0}\right)$

$\Rightarrow t = \frac{2\log\left(\frac{y}{y_0}\right)}{\log\left(\frac{11}{10}\right)} \dots (4)$ **[1 Mark]**

Now, let the time when the number of bacteria increases from 100000 to 200000 be t_1.

$\Rightarrow y = 2y_0$ at $t = t_1$

From equation (4), we get:

$$t_1 = \frac{2\log\left(\frac{y}{y_0}\right)}{\log\left(\frac{11}{10}\right)} = \frac{2\log 2}{\log\left(\frac{11}{10}\right)}$$

Hence, in $\frac{2\log 2}{\log\left(\frac{11}{10}\right)}$ hours the number of bacteria increases from 100000 to 200000. **[1 Mark]**

23. The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ is **[2 Marks]**

(A) $e^{x} + e^{-y} = C$

(B) $e^{x} + e^{y} = C$

(C) $e^{-x} + e^{y} = C$

(D) $e^{-x} + e^{-y} = C$

Solution:

Given differential equation is $\frac{dy}{dx} = e^{x+y} = e^{x} \cdot e^{y}$

$\Rightarrow \frac{dy}{e^{y}} = e^{x}dx$

$\Rightarrow e^{-y}dy = e^{x}dx$

Integrating both sides, we get:

$\int e^{-y}\,dy = \int e^{x}\,dx$ **[1 Mark]**

$\Rightarrow -e^{-y} = e^{x} + k$

$\Rightarrow e^{x} + e^{-y} = -k$

$\Rightarrow e^{x} + e^{-y} = c \dots (c = -k)$

Therefore, the general solution of the given differential equation is $e^{x} + e^{-y} = c \dots (c = -k)$

Hence, the correct answer is A. **[1 Mark]**

Exercise 9.5

In each of the Exercises 1 to 10, show that the given differential equation is homogeneous and solve each of them.

1. $(x^2 + xy)dy = (x^2 + y^2)dx$ **[6 Marks]**

Solution:

Given differential equation is $(x^2 + xy)dy = (x^2 + y^2)dx$

It can be written as $\frac{dy}{dx} = \frac{x^2+y^2}{x^2+xy}$

Let $F(x, y) = \frac{x^2+y^2}{x^2+xy}$

Now, $F(\lambda x, \lambda y) = \frac{(\lambda x)^2+(\lambda y)^2}{(\lambda x)^2+(\lambda x)(\lambda y)} = \frac{x^2+y^2}{x^2+xy} = \lambda^0 \cdot F(x, y)$

This shows that the given equation is a homogeneous equation. **[1 Mark]**

To solve it, we make the substitution as:

$y = vx$

Differentiating both sides with respect to x, we get:

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of v and $\frac{dy}{dx}$ in the given equation, we get:

$v + x\frac{dv}{dx} = \frac{x^2+(vx)^2}{x^2+x(vx)}$ **[1 Mark]**

$\Rightarrow v + x\frac{dv}{dx} = \frac{1+v^2}{1+v}$

$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{1+v} - v = \frac{(1+v^2)-v(1+v)}{1+v}$

$\Rightarrow x\frac{dv}{dx} = \frac{1-v}{1+v}$

$\Rightarrow \left(\frac{1+v}{1-v}\right) = dv = \frac{dx}{x}$

$\Rightarrow \left(\frac{2-1+v}{1-v}\right) dv = \frac{dx}{x}$

$\Rightarrow \left(\frac{2}{1-v} - 1\right) dv = \frac{dx}{x}$ **[1 Mark]**

Integrating both sides, we get:

$-2\log(1-v) - v = \log x - \log k$

$\Rightarrow v = -2\log(1-v) - \log x + \log k$

$\Rightarrow v = \log\left[\frac{k}{x(1-v)^2}\right]$

$\Rightarrow \frac{y}{x} = \log\left[\frac{k}{x\left(1-\frac{y}{x}\right)^2}\right]$

$\Rightarrow \frac{y}{x} = \log\left[\frac{kx}{(x-y)^2}\right]$

$\Rightarrow \frac{kx}{(x-y)^2} = e^{\frac{y}{x}}$

$\Rightarrow (x-y)^2 = kxe^{\frac{y}{x}}$

Hence, the required solution of the given differential equation is $(x-y)^2 = kxe^{\frac{y}{x}}$ **[1 Mark]**

2. $y' = \frac{x+y}{x}$ **[4 Marks]**

Solution:

The given differential equation is $y' = \frac{x+y}{x}$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x} \ldots (1)$$

Let $F(x, y) = \frac{x+y}{x}$.

Now, $F(\lambda x, \lambda y) = \frac{\lambda x+\lambda y}{\lambda x} = \frac{x+y}{x} = \lambda^0 F(x, y)$

Thus, the given equation is a homogeneous equation. **[1 Mark]**

To solve it, we make the substitution as:

$y = vx$

Differentiating both sides with respect to x, we get:

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{x + vx}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = 1 + v$$

$$x\frac{dv}{dx} = 1$$

$\Rightarrow dv = \frac{dx}{x}$ **[2 Marks]**

Integrating both sides, we get:

$v = \log x + C$

$$\Rightarrow \frac{y}{x} = \log x + C$$

$\Rightarrow y = x\log x + Cx$

Hence, the required solution of the given differential equation is $y = x\log x + Cx$ **[1 Mark]**

3. $(x - y)dy - (x + y)dx = 0$ **[4 Marks]**

Solution:

Given differential equation is $(x-y)dy-(x+y)dx=0$

$$\Rightarrow \frac{dy}{dx}=\frac{x+y}{x-y} \dots (1)$$

Let $F(x,y)=\frac{x+y}{x-y}$

$$\therefore F(\lambda x,\lambda y)=\frac{\lambda x+\lambda y}{\lambda x-\lambda y}=\frac{x+y}{x-y}=\lambda^0\cdot F(x,y)$$

Thus, the given differential equation is a homogeneous equation. **[1 Mark]**

To solve it, we make the substitution as:

$y=vx$

$\Rightarrow \frac{d}{dx}(y)=\frac{d}{dx}(vx)$

$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

Substituting the values of y and $\frac{dy}{dx}$

$$v + x\frac{dv}{dx} = \frac{x + vx}{x - vx} = \frac{1 + v}{1 - v}$$

$$x\frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+v-v(1-v)}{1-v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1 + v^2}{1 - v}$$

$$\Rightarrow \frac{1 - v}{(1 + v^2)}dv = \frac{dx}{x}$$

$$\Rightarrow \left(\frac{1}{1+v^2} - \frac{v}{1-v^2}\right)dv = \frac{dx}{x}$$

[1 Mark]

Integrating both sides, we get:

$$\tan^{-1}v - \frac{1}{2}\log(1 + v^2) = \log x + C$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\log\left[1 + \left(\frac{y}{x}\right)^2\right] = \log x + C$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\log\left(\frac{x^2 + y^2}{x^2}\right) = \log x + C$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}[\log(x^2 + y^2) - \log x^2] = \log x + C$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\log(x^2 + y^2) + \mathrm{C}$$

Hence, the required solution of the given differential equation is

$$\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\log(x^2 + y^2) + \mathrm{C}$$

[2 Marks]

4. $(x^2 - y^2)dx + 2xydy = 0$ **[4 Marks]**

Solution:

The given differential equation is $(x^2 - y^2)dx + 2xydy = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{-(x^2 - y^2)}{2xy} \dots (1)$$

Let $F(x, y) = \frac{-(x^2-y^2)}{2xy}$

$$\therefore F(\lambda x, \lambda y) = \left[\frac{(\lambda x)^2 - (\lambda y)^2}{2(\lambda x)(\lambda y)}\right] = \frac{-(x^2 - y^2)}{2xy} = \lambda^0 \cdot F(x, y)$$

Therefore, the given differential equation is a homogeneous equation. **[1 Mark]**

To solve it, we make the substitution as:

$y = vx$

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ **[1 Mark]**

Substituting the values of y and $\frac{dy}{dx}$

$$v + x\frac{dv}{dx} = -\left[\frac{x^2 - (vx)^2}{2x \cdot (vx)}\right]$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v = \frac{v^2 - 1 - 2v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{(1 + v^2)}{2v}$$

$\Rightarrow \frac{2v}{1+v^2}dv = -\frac{dx}{x}$ **[1 Mark]**

Integrating both sides, we get:

$$\log(1 + v^2) = -\log x + \log C = \log\frac{C}{x}$$

$$\Rightarrow 1 + v^2 = \frac{C}{x}$$

$$\Rightarrow \left[1 + \frac{y^2}{x^2}\right] = \frac{C}{x}$$

$$\Rightarrow x^2 + y^2 = Cx$$

Hence, the required solution of the given differential equation is $x^2 + y^2 = Cx$ **[1 Mark]**

5. $x^2\frac{dy}{dx} - x^2 - 2y^2 + xy$ **[4 Marks]**

Solution:

Given differential equation is $x^2\frac{dy}{dx}-x^2-2y^2+xy$

$$\frac{dy}{dx}=\frac{x^2-2y^2+xy}{x^2}\ldots(1)$$

Let $F(x,y)=\frac{x^2-2y^2+xy}{x^2}$

$\therefore F(\lambda x,\lambda y)=\frac{(\lambda x)^2-2(\lambda y)^2+(\lambda x)(\lambda y)}{(\lambda x)^2}=\frac{x^2-2y^2+xy}{x^2}=\lambda^0\cdot F(x,y)$ **[1 Mark]**

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$y=vx$

$\Rightarrow\frac{dy}{dx}=v+x\frac{dv}{dx}$ **[1 Mark]**

Substituting the values of y and $\frac{dy}{dx}$

$$v+x\frac{dv}{dx}=\frac{x^2-2(vx)^2+x\cdot(vx)}{x^2}$$

$$\Rightarrow v+x\frac{dv}{dx}=1-2v^2+v$$

$$\Rightarrow x\frac{dv}{dx}=1-2v^2$$

$$\Rightarrow\frac{dv}{1-2v^2}=\frac{dx}{x}$$

$$\Rightarrow\frac{1}{2}\cdot\frac{dv}{\frac{1}{2}-v^2}=\frac{dx}{x}$$

$\Rightarrow\frac{1}{2}\cdot\left[\frac{dv}{\left(\frac{1}{\sqrt{2}}\right)^2-v^2}\right]=\frac{dx}{x}$ **[1 Mark]**

Integrating both sides, we get:

$$\frac{1}{2}\cdot\frac{1}{2\times\frac{1}{\sqrt{2}}}\log\left|\frac{\frac{1}{\sqrt{2}}+v}{\sqrt{2}-v}\right|=\log|x|+C$$

$$\Rightarrow\frac{1}{2\sqrt{2}}\log\left|\frac{\frac{1}{\sqrt{2}}+\frac{y}{x}}{\frac{1}{\sqrt{2}}-\frac{y}{x}}\right|=\log|x|+\mathrm{C}$$

$\Rightarrow \frac{1}{2\sqrt{2}}\log\left|\frac{x+\sqrt{2}y}{x-\sqrt{2}y}\right| = \log|x| + C$ **[1 Mark]**

Hence, the required solution for the given differential equation is $\frac{1}{2\sqrt{2}}\log\left|\frac{x+\sqrt{2}y}{x-\sqrt{2}y}\right| = \log|x| + C$

6. $xdy - ydx = \sqrt{x^2+y^2}dx$ **[4 Marks]**

Solution:

Given differential equation is $xdy - ydx = \sqrt{x^2+y^2}dx$

$\Rightarrow xdy = \left[y + \sqrt{x^2+y^2}\right]dx$

$$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x^2} \dots (1)$$

Let $F(x,y) = \frac{y+\sqrt{x^2+y^2}}{x^2}$.

$$\therefore F(\lambda x, \lambda y) = \frac{\lambda x + \sqrt{(\lambda x)^2 + (\lambda y)^2}}{\lambda x} = \frac{y+\sqrt{x^2+y^2}}{x} = \lambda^0 \cdot F(x,y)$$

Therefore, the given differential equation is a homogeneous equation. **[1 Mark]**

To solve it, we make the substitution as:

$y = vx$

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ **[1 Mark]**

Substituting the values of v and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2+(vx)^2}}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = v + \sqrt{1+v^2}$$

$\Rightarrow \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$ **[1 Mark]**

Integrating both sides, we get:

$\log\left|v+\sqrt{1+v^2}\right| = \log|x| + \log C$

$\Rightarrow \log\left|\frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}}\right| = \log|Cx|$

$\Rightarrow \log\left|\frac{y+\sqrt{x^2+y^2}}{x}\right| = \log|Cx|$

$\Rightarrow y+\sqrt{x^2+y^2} = Cx^2$

Hence, the required solution of the given differential equation is $y+\sqrt{x^2+y^2} = Cx^2$ **[1 Mark]**

7. $\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}ydx = \left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}xdy$ **[4 Marks]**

Solution:

Given differential equation is $\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}ydx = \left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}xdy$

$\frac{dy}{dx} = \frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x} \ldots (1)$

Let $F(x,y) = \frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x}$

$\therefore F(\lambda x,\lambda y) = \frac{\left\{\lambda x\cos\left(\frac{\lambda y}{\lambda x}\right)+\lambda y\sin\left(\frac{\lambda y}{\lambda x}\right)\right\}\lambda y}{\left\{\lambda y\sin\left(\frac{\lambda y}{\lambda x}\right)-\lambda x\sin\left(\frac{\lambda y}{\lambda x}\right)\right\}\lambda x}$

$= \frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x}$

$= \lambda^0 \cdot F(F,y)$

Therefore, the given differential equation is a homogeneous equation. **[1 Mark]**

To solve it, we make the substitution as:

$y = vx$

$\Rightarrow \frac{dy}{dx} = v + x = \frac{dv}{dx}$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$v + x\frac{dv}{dx} = \frac{(x\cos v + vx\sin v)\cdot vx}{(vx\sin v - x\cos v)\cdot x}$ **[1 Mark]**

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v - v^2\sin v + v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \left[\frac{v\sin v - \cos v}{v\cos v}\right]dv = \frac{2dx}{x}$$

$\Rightarrow \left(\tan v - \frac{1}{v}\right)dv = \frac{2dx}{x}$ **[1 Mark]**

Integrating both sides, we get:

$$\log(\sec v) - \log v = 2\log x + \log C$$

$$\Rightarrow \log\left(\frac{\sec v}{v}\right) = \log(Cx^2)$$

$$\Rightarrow \left(\frac{\sec v}{v}\right) = Cx^2$$

$$\Rightarrow \sec v = Cx^2 v$$

$$\Rightarrow \sec\left(\frac{y}{x}\right) = C \cdot x^2 \cdot \frac{y}{x}$$

$$\Rightarrow \sec\left(\frac{y}{x}\right) = Cxy$$

$$\Rightarrow \cos\left(\frac{y}{x}\right) = \frac{1}{Cxy} = \frac{1}{C}\cdot\frac{1}{xy}$$

$$\Rightarrow xy\cos\left(\frac{y}{x}\right) = k \ \left(k = \frac{1}{C}\right)$$

Hence, the required solution of the given differential equation is $xy\cos\left(\frac{y}{x}\right) = k$. **[1 Mark]**

8. $x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$ **[4 Marks]**

Solution:

Give differential equation is $x\frac{dy}{dx}-y+x\sin\left(\frac{y}{x}\right)=0$

$$\Rightarrow x\frac{dy}{dx}=y-x\sin\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{y-x\sin\left(\frac{y}{x}\right)}{x}..(1)$$

Let $F(x,y)=\frac{y-x\sin\left(\frac{y}{x}\right)}{x}$

$$\therefore F(\lambda x,\lambda y)=\frac{\lambda y-\lambda x\sin\left(\frac{\lambda y}{\lambda x}\right)}{\lambda x}=\frac{y-x\sin\left(\frac{y}{x}\right)}{x}=\lambda^0\cdot F(x,y)$$

Therefore, the given differential equation is a homogeneous equation. **[1 Mark]**

To solve it, we make the substitution as:

$y=vx$

$$\Rightarrow \frac{d}{dx}(y)=\frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$v+x\frac{dv}{dx}=\frac{vx-x\sin v}{x}$ **[1 Mark]**

$$\Rightarrow v+x\frac{dv}{dx}=v-\sin v$$

$$\Rightarrow -\frac{dv}{\sin v}=\frac{dx}{x}$$

$\Rightarrow \operatorname{cosec} v\, dv=-\frac{dx}{x}$ **[1 Mark]**

Integrating both sides, we get:

$$\log|\operatorname{cosec} v-\cot v|=-\log x+\log C=\log\frac{C}{x}$$

$$\Rightarrow \operatorname{cosec}\left(\frac{y}{x}\right)-\cot\left(\frac{y}{x}\right)=\frac{C}{x}$$

$$\Rightarrow \frac{1}{\sin\left(\frac{y}{x}\right)}-\frac{\cos\left(\frac{y}{x}\right)}{\sin\left(\frac{y}{x}\right)}=\frac{C}{x}$$

$\Rightarrow x\left[1-\cos\left(\frac{y}{x}\right)\right]=C\sin\left(\frac{y}{x}\right)$

Hence, the required solution of the given differential equation is $x\left[1-\cos\left(\frac{y}{x}\right)\right]=C\sin\left(\frac{y}{x}\right)$ **[1 Mark]**

9. $ydx+x\log\left(\frac{y}{x}\right)dy-2xdy=0$ **[6 Marks]**

Solution:

Given differential equation is $ydx+x\log\left(\frac{y}{x}\right)dy-2xdy=0$

$\Rightarrow ydx=\left[2x-x\log\left(\frac{y}{x}\right)\right]dy$

$\Rightarrow \frac{dy}{dx}=\frac{y}{2x-x\log\left(\frac{y}{x}\right)}\ldots(1)$

Let $F(x,y)=\frac{y}{2x-x\log\left(\frac{y}{x}\right)}$

$\therefore F(\lambda x,\lambda y)=\frac{\lambda y}{2(\lambda x)-(\lambda x)\log\left(\frac{\lambda y}{\lambda x}\right)}=\frac{y}{2x-\log\left(\frac{y}{x}\right)}=\lambda^0\cdot F(x,y)$

Therefore, the given differential equation is a homogeneous equation. **[1 Mark]**

To solve it, we make the substitution as:

$y=vx$

$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}(vx)$

$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

Substituting the values of y and $\frac{dy}{dx}$

$v+x\frac{dv}{dx}=\frac{vx}{2x-x\log y}$ **[1 Mark]**

$\Rightarrow v+x\frac{dv}{dx}=\frac{v}{2-\log v}$

$\Rightarrow x\frac{dv}{dx}=\frac{v}{2-\log v}-v$

$$\Rightarrow x\frac{dv}{dx}=\frac{v-2v+v\log v}{2-\log v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v\log v-v}{2-\log v}$$

$$\Rightarrow \frac{2-\log v}{v(\log v-1)}dv=\frac{dx}{x}$$

$$\Rightarrow \left[\frac{1+(1-\log v)}{v(\log v-1)}\right]dv=\frac{dx}{x}$$

$$\Rightarrow \left[\frac{1}{v(\log v-1)}-\frac{1}{v}\right]dv=\frac{dx}{x}$$

[1 Mark]

Integrating both sides, we get:

$$\int\frac{1}{v(\log v-1)}dv-\int\frac{1}{v}dv=\int\frac{1}{x}dx$$

$$\Rightarrow \int\frac{dv}{v(\log v-1)}-\log v=\log x+\log C\ \ldots(2)$$

$$\Rightarrow \text{Let } \log v-1=t$$

$$\Rightarrow \frac{d}{dv}(\log v-1)=\frac{dt}{dv}$$

$$\Rightarrow \frac{1}{v}=\frac{dt}{dv}$$

$$\Rightarrow \frac{dv}{v}=dt$$

[1 Mark]

Therefore, equation (1) becomes:

$$\Rightarrow \int\frac{dt}{t}-\log v=\log x+\log C$$

$$\Rightarrow \log t-\log\left(\frac{y}{x}\right)=\log(Cx)$$

$$\Rightarrow \log\left[\log\left(\frac{y}{x}\right)-1\right]-\log\left(\frac{y}{x}\right)=\log(Cx)$$

$$\Rightarrow \log\left[\frac{\log\left(\frac{y}{x}\right)-1}{\frac{y}{x}}\right]=\log(Cx)$$

$$\Rightarrow \frac{x}{y}\left[\log\left(\frac{y}{x}\right)-1\right]=Cx$$

$$\Rightarrow \log\left(\frac{y}{x}\right)-1=Cy$$

Hence, the required solution of the given differential equation is $\log\left(\frac{y}{x}\right)-1=Cy$ **[2 Marks]**

10. $\left(1+e^{\frac{x}{y}}\right)dx+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$ **[4 Marks]**

Solution:

Given differential equation is $\left(1+e^{\frac{x}{y}}\right)dx+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$

$\Rightarrow \left(1+e^{\frac{x}{y}}\right)dx=-e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy$

$\Rightarrow \frac{dx}{dy}=\frac{-e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{1+e^{\frac{x}{y}}} \dots (1)$

Let $F(x,y)=\frac{-e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{1+e^{\frac{x}{y}}}$

$\therefore F(\lambda x,\lambda y)=\frac{-e^{\frac{\lambda x}{\lambda y}}\left(1-\frac{\lambda x}{\lambda y}\right)}{1+e^{\frac{\lambda x}{\lambda y}}}=\frac{-e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{1+e^{\frac{x}{y}}}=\lambda^0\cdot F(x,y)$

Therefore, the given differential equation is a homogeneous equation. **[1 Mark]**

To solve it, we make the substitution as:

$x=vy$

$\Rightarrow \frac{d}{dy}(x)=\frac{d}{dy}(vy)$

$\Rightarrow \frac{dx}{dy}=v+y\frac{dv}{dy}$

Substituting the values of x and $\frac{dx}{dy}$

$v+y\frac{dv}{dy}=\frac{-e^{v}(1-v)}{1+e^{v}}$ **[1 Mark]**

$\Rightarrow y\frac{dv}{dy}=\frac{-e^{v}+ve^{v}}{1+e^{v}}-v$

$\Rightarrow y\frac{dv}{dy}=\frac{-e^{v}+ve^{v}-v-ve^{v}}{1+e^{v}}$

$\Rightarrow y\frac{dv}{dy}=-\left[\frac{v+e^{v}}{1+e^{v}}\right]$

$\Rightarrow \left[\frac{1+e^v}{v+e^v}\right] dv = -\frac{dy}{y}$ **[1 Mark]**

Integrating both sides, we get:

$\Rightarrow \log(v + e^v) = -\log y + \log C = \log\left(\frac{C}{y}\right)$

$\Rightarrow \left[\frac{x}{y} + e^{\frac{x}{y}}\right] = \frac{C}{y}$

$\Rightarrow x + ye^{\frac{x}{y}} = C$

Hence, the required solution of the given differential equation is $x + ye^{\frac{x}{y}} = C$ **[1 Mark]**

For each of the differential equations in Exercises from 11 to 15, find the particular solution satisfying the given condition:

11. $(x + y)dy + (x - y)dy = 0; y = 1$ when $x = 1$ **[6 Marks]**

Solution:

Given differential equation is $(x + y)dy + (x - y)dy = 0$

$\Rightarrow (x + y)dy = -(x - y)dx$

$\Rightarrow \frac{dy}{dx} = \frac{-(x-y)}{x+y} \dots (1)$

Let $F(x, y) = \frac{-(x-y)}{x+y}$

$\therefore F(\lambda x, \lambda y) = \frac{-(\lambda x - \lambda y)}{\lambda x - \lambda y} = \frac{-(x-y)}{x+y} = \lambda^0 \cdot F(x, y)$

Therefore, the given differential equation is a homogeneous equation. **[1 Mark]**

To solve it, we make the substitution as:

$y = vx$

$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$v + x\frac{dv}{dx} = \frac{-(x-vx)}{x+vx}$ **[1 Mark]**

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v-1}{v+1}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v-1}{v+1} - v = \frac{v-1-v(v+1)}{v+1}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v-1-v^2-v}{v+1} = \frac{-(1+v^2)}{v+1}$$

$$\Rightarrow \frac{(v+1)}{1+v^2}dv = -\frac{dx}{x}$$

$\Rightarrow \left[\frac{v}{1+v^2} + \frac{1}{1+v^2}\right]dv = -\frac{dx}{x}$ **[1 Mark]**

Integrating both sides, we get:

$$\frac{1}{2}\log(1+v^2) + \tan^{-1}v = -\log x + k$$

$$\Rightarrow \log(1+v^2) + 2\tan^{-1}v = -2\log x + 2k$$

$$\Rightarrow \log[(1+v^2)\cdot x^2] + 2\tan^{-1}v = 2k$$

$$\Rightarrow \log\left[\left(1+\frac{y^2}{x^2}\right)\cdot x^2\right] + 2\tan^{-1}\frac{y}{x} = 2k$$

$\Rightarrow \log(x^2+y^2) + 2\tan^{-1}\frac{y}{x} = 2k \ldots (2)$ **[1 Mark]**

Now, $y = 1$ at $x = 1$

$$\Rightarrow \log 2 + 2\tan^{-1}1 = 2k$$

$$\Rightarrow \log 2 + 2 \times \frac{\pi}{4} = 2k$$

$\Rightarrow \frac{\pi}{2} + \log 2 = 2k$ **[1 Mark]**

Substituting the value of $2k$ in equation (2), we get:

$$\log(x^2+y^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{2} + \log 2$$

Hence, the required solution of the given differential equation is $\log(x^2+y^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{2} + \log 2$

[1 Mark]

12. $x^2dy + (xy + y^2)dx = 0; y = 1$ when $x = 1$ **[6 Marks]**

Solution:

$x^2dy + (xy + y^2)dx = 0$

$\Rightarrow x^2dy = -(xy + y^2)dx$

$\Rightarrow \frac{dy}{dx} = \frac{-(xy + y^2)}{x^2} \dots (1)$

Let $F(x, y) = \frac{-(xy+y^2)}{x^2}$

$\therefore F(\lambda x, \lambda y) = \frac{[\lambda x \cdot \lambda y + (\lambda y)^2]}{(\lambda x)^2} = \frac{-(xy + y^2)}{x^2} = \lambda^0 \cdot F(x, y)$

Therefore, the given differential equation is a homogeneous equation. **[1 Mark]**

To solve it, we make the substitution as:

$y = vx$

$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting the values of y and $\frac{dy}{dx}$

$v + x\frac{dv}{dx} = \frac{-[x \cdot vx + (vx)^2]}{x^2} = -v - v^2$ **[1 Mark]**

$\Rightarrow x\frac{dv}{dx} = -v^2 - 2v = -v(v + 2)$

$\Rightarrow \frac{dv}{v(v + 2)} = -\frac{dx}{x}$

$\Rightarrow \frac{1}{2}\left[\frac{(v + 2) - v}{v(v + 2)}\right]dv = -\frac{dx}{x}$

$\Rightarrow \frac{1}{2}[\frac{1}{v} - \frac{1}{v+2}]dv = -\frac{dx}{x}$ **[1 Mark]**

Integrating both sides, we get:

$\frac{1}{2}[\log v - \log(v + 2)] = -\log x + \log C$

$\Rightarrow \frac{1}{2}\log\left(\frac{v}{v+2}\right) = \log\frac{C}{x}$

$\Rightarrow \frac{v}{v+2} = \left(\frac{C}{x}\right)^2$

$\Rightarrow \frac{\frac{y}{x}}{\frac{y}{x}+2} = \left(\frac{C}{x}\right)^2$

$\Rightarrow \frac{y}{y+2x} = \frac{C^2}{x^2}$

$\Rightarrow \frac{x^2y}{y+2x} = C^2 \ldots (2)$ **[1 Mark]**

Now, $y = 1$ at $x = 1$

$\Rightarrow \frac{1}{1+2} = C^2$

$\Rightarrow C^2 = \frac{1}{3}$ **[1 Mark]**

Substituting $C^2 = \frac{1}{3}$

$\frac{x^2y}{y+2x} = \frac{1}{3}$

$\Rightarrow y + 2x = 3x^2y$

Hence, the required solution of the given differential equation is $y + 2x = 3x^2y$ **[1 Mark]**

13. $\left[x\sin^2\left(\frac{x}{y} - y\right)\right]dx + xdy = 0; y\frac{\pi}{4}$ when $x = 1$ **[6 Marks]**

Solution:

Given differential equation is $\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + xdy = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-\left[x\sin^2\left(\frac{y}{x}\right) - y\right]}{x} \ldots (1)$

Let $F(x, y) = \frac{-\left[x\sin^2\left(\frac{y}{x}\right) - y\right]}{x}$

$$\therefore F(\lambda x, \lambda y) = \frac{-\left[\lambda x \cdot \sin^2\left(\frac{\lambda x}{\lambda y}\right) - \lambda y\right]}{\lambda x} = \frac{-\left[x\sin^2\left(\frac{y}{x}\right) - y\right]}{x} = \lambda^0 \cdot F(x, y)$$

Therefore, the given differential equation is a homogeneous equation. **[1 Mark]**

To solve this differential equation, we make the substitution as:

$y = vx$

$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$

$\Rightarrow \frac{dy}{dx} = v + x = \frac{dv}{dx}$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$v + x\frac{dv}{dx} = \frac{-[x\sin^2 v - vx]}{x}$ **[1 Mark]**

$\Rightarrow v + x\frac{dv}{dx} = -[\sin^2 v - v] = v - \sin^2 v$

$\Rightarrow x\frac{dv}{dx} = -\sin^2 v$

$\Rightarrow \frac{dv}{\sin^2 v} = -\sin^2 v$

$\Rightarrow \text{cosec}^2 v dv = -\frac{dx}{x}$ **[1 Mark]**

Integrating both sides, we get:

$-\cot v = -\log|x| - C$

$\Rightarrow \cot v = \log|x| + C$

$\Rightarrow \cot\left(\frac{y}{x}\right) = \log|x| + \log C$

$\Rightarrow \cot\left(\frac{y}{x}\right) = \log|Cx| \ldots (2)$

Now, $y = \frac{\pi}{4}$ at $x = 1$

$\Rightarrow \cot\left(\frac{\pi}{4}\right) = \log|C|$

$\Rightarrow 1 = \log C$

$\Rightarrow C = e^1 = e$ **[2 Marks]**

Substituting $C = e$ in equation (2), we get:

$\cot\left(\frac{y}{x}\right) = \log|ex|$

Hence, the required solution of the given differential equation is $\cot\left(\frac{y}{x}\right) = \log|ex|$ **[1 Mark]**

14. $\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\left(\frac{y}{x}\right) = 0; y = 0$ when $x = 1$ **[6 Marks]**

Solution:

Given differential equation is $\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\left(\frac{y}{x}\right) = 0$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \operatorname{cosec}\left(\frac{y}{x}\right) \ldots (1)$

Let $F(x, y) = \frac{y}{x} - \operatorname{cosec}\left(\frac{y}{x}\right)$

$\therefore F(\lambda x, \lambda y) = \frac{\lambda y}{\lambda x} - \operatorname{cosec}\left(\frac{\lambda y}{\lambda x}\right)$

$\Rightarrow F(\lambda x, \lambda y) = \frac{y}{x} - \operatorname{cosec}\left(\frac{y}{x}\right) = F(x, y) = \lambda^0 \cdot F(x, y)$

Therefore, the given differential equation is a homogeneous equation. **[1 Mark]**

To solve it, we make the substitution as:

$y = vx$

$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$v + x\frac{dv}{dx} = v - \operatorname{cosec} v$ **[1 Mark]**

$\Rightarrow -\frac{dv}{\operatorname{cosec} v} = -\frac{dx}{x}$

$\Rightarrow -\sin v\, dv = \frac{dx}{x}$ **[1 Mark]**

Integrating both sides, we get:

$\cos v = \log x + \log C = \log|Cx|$

$\Rightarrow \cos\left(\frac{y}{x}\right) = \log|Cx| \ldots (2)$ **[1 Mark]**

This is the required solution of the given differential equation.

Now, $y = 0$ at $x = 1$.

$\Rightarrow \cos(0) = \log C$

$\Rightarrow 1 = \log C$

$\Rightarrow C = e^1 = e$ **[1 Mark]**

Substituting $C = e$ in equation (2), we get:

$\cos\left(\frac{y}{x}\right) = \log|(ex)|$

Hence, the required solution of the given differential equation is $\cos\left(\frac{y}{x}\right) = \log|(ex)|$ **[1 Mark]**

15. $2xy + y^2 - 2x^2\frac{dy}{dx} = 0; y = 2$ when $x = 1$ **[6 Marks]**

Solution:

Given differential equation is $2xy + y^2 - 2x^2\frac{dy}{dx} = 0$

$\Rightarrow 2x^2\frac{dy}{dx} = 2xy + y^2$

$\Rightarrow \frac{dy}{dx} = \frac{2xy + y^2}{2x^2} \ldots (1)$

Let $F(x, y) = \frac{2xy+y^2}{2x^2}$

$\therefore F(\lambda x, \lambda y) = \frac{2(\lambda x)(\lambda y) + (\lambda y)^2}{2(\lambda x)^2} = \frac{2xy + y^2}{2x^2} = \lambda^0 \cdot F(x, y)$

Therefore, the given differential equation is a homogeneous equation. **[1 Mark]**

To solve it, we make the substitution as:

$y = vx$

$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting the value of y and $\frac{dy}{dx}$ in equation (1), we get:

$v + x\frac{dv}{dx} = \frac{2x(vx)+(vx)^2}{2x^2}$ **[1 Mark]**

$\Rightarrow v + x\frac{dv}{dx} = \frac{2v + v^2}{2}$

$\Rightarrow v + x\frac{dv}{dx} = v + \frac{v^2}{2}$

$\Rightarrow \frac{2}{v^2}dv = \frac{dx}{x}$ **[1 Mark]**

Integrating both sides, we get:

$2 \cdot \frac{v^{-2+1}}{-2+1} = \log|x| + C$

$\Rightarrow -\frac{2}{v} = \log|x| + C$

$\Rightarrow -\frac{2}{\frac{y}{x}} = \log|x| + C$

$\Rightarrow -\frac{2x}{y} = \log|x| + C \ldots(2)$ **[1 Mark]**

Now, $y = 2$ at $x = 1$

$\Rightarrow -1 = \log(1) + C$

$\Rightarrow C = -1$ **[1 Mark]**

Substituting $C = -1$ in equation (2), we get:

$-\frac{2x}{y} = \log|x| - 1$

$\Rightarrow \frac{2x}{y} = 1 - \log|x|$

$\Rightarrow y = \frac{2x}{1 - \log|x|}, (x \neq 0, x \neq e)$

Hence, the required solution of the given differential equation is $y = \frac{2x}{1-\log|x|}$ **[1 Mark]**

16. A homogeneous differential equation of the form $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$ can be solved by making the substitution.
[1 Mark]

(A) $y = vx$

(B) $v = yx$

(C) $x = vy$

(D) $x = v$

Solution:

For solving the homogeneous equation of the form $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$, we need to make the substitution as

$x = vy$.

Hence, the correct answer is C. **[1 Mark]**

17. Which of the following is a homogeneous differential equation? **[2 Marks]**

(A) $(4x + 6y + 5)dy - (3y + 2x + 4)dx = 0$

(B) $(xy)dx - (x^3 + y^3)dy = 0$

(C) $(x^3 + 2y^2)dx + 2xydy = 0$

(D) $y^2dx + (x^2 - xy^2 - y^2)dy = 0$

Solution:

Function $F(x, y)$ is said to be the homogenous function of degree n, if $F(\lambda x, \lambda y) = \lambda^n F(x, y)$ for any non-zero constant (λ).

Consider the equation given in alternative D:

$y^2dx + (x^2 - xy - y^2)dy = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-y^2}{x^2-xy-y^2} = \frac{y^2}{y^2+xy-x^2}$ **[1 Mark]**

Let $F(x, y) = \frac{y^2}{y^2+xy-x^2}$

$$\Rightarrow F(\lambda x, \lambda y) = \frac{(\lambda y)^2}{(\lambda y)^2 + (\lambda x)(\lambda y) - (\lambda x)^2}$$

$$= \frac{\lambda^2 y^2}{\lambda^2 (y^2 + xy - x^2)}$$

$$== \lambda^0 \left(\frac{y^2}{y^2 + xy - x^2} \right)$$

$$= \lambda^0 \cdot F(x, y)$$

Hence, the differential equation given in alternative D is a homogenous equation. **[1 Mark]**

Exercise 9.6

For each of the differential equations given in Exercises 1 to 12, find the general solution:

1. $\frac{dy}{dx} + 2y = \sin x$ **[4 Marks]**

Solution:

Given differential equation is $\frac{dy}{dx} + 2y = \sin x$

This is in the form of $\frac{dy}{dx} + Py = Q$ (where $P = 2$ and $Q = \sin x$)

Now,=$e^{\int P dx} = e^{\int 2dx} = e^{2x}$

The solution of the given differential equation is given by the relation,

$$y(I.F.) = \int (Q \times I.F.)\, dx + C$$

$\Rightarrow ye^{2x} = \int \sin x \cdot e^{2x} dx + C \ldots (1)$ **[1 Mark]**

Let $I = \int \sin x \cdot e^{2x}$.

$$\Rightarrow I = \sin x \cdot \int e^{2x}\, dx - \int \left(\frac{d}{dx}(\sin x) \cdot \int e^{2x}\, dx \right) dx$$

$$\Rightarrow I = \sin x \cdot \frac{e^{2x}}{2} - \int \left(\cos x \cdot \frac{e^{2x}}{2} \right) dx$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{1}{2} \left[\cos x \cdot \int e^{2x} - \int \left(\frac{d}{dx}(\cos x) \cdot \int e^{2x}\, dx \right) dx \right]$$

$$\Rightarrow I = \frac{e^{2x}\sin x}{2} - \frac{1}{2}\left[\cos x \cdot \frac{e^{2x}}{2} - \int \left[(-\sin x)\cdot \frac{e^{2x}}{2}\right] dx\right]$$

$$\Rightarrow I = \frac{e^{2x}\sin x}{2} - \frac{e^{2x}\cos x}{4} - \frac{1}{4}\int (\sin x \cdot e^{2x})\, dx$$

$$\Rightarrow I = \frac{e^{2x}}{4}(2\sin x - \cos x) - \frac{1}{4}I$$

$$\Rightarrow \frac{5}{4}I = \frac{e^{2x}}{4}(2\sin x - \cos x)$$

$\Rightarrow I = \frac{e^{2x}}{5}(2\sin x - \cos x)$ **[2 Marks]**

Therefore, equation (1) becomes:

$$ye^{2x} = \frac{e^{2x}}{5}(2\sin x - \cos x) + C$$

$$\Rightarrow y = \frac{1}{5}(2\sin x - \cos x) + Ce^{-2x}$$

Hence, the required general solution of the given differential equation is

$y = \frac{1}{5}(2\sin x - \cos x) + Ce^{-2x}$ **[1 Mark]**

2. $\frac{dy}{dx} + 3y = e^{-2x}$ **[2 Marks]**

Solution:

The given differential equation is $\frac{dy}{dx} + 3y = e^{-2x}$

This is in the form of $\frac{dy}{dx} + Py = Q$ (where $P = 3$ and $Q = e^{-2x}$)

Now, $I.F = e^{\int P dx} = e^{\int 3 dx} = e^{3x}$ **[1 Mark]**

The solution of the given differential equation is given by the relation,

$$y(\text{I.F.}) = \int (\text{Q} \times \text{I.F.})\, dx + C$$

$$\Rightarrow ye^{3x} = \int (e^{-2x} \times e^{3x}) + C$$

$$\Rightarrow ye^{3x} = \int e^{x}\, dx + C$$

$\Rightarrow ye^{3x} = e^{x} + \mathrm{C}$

$\Rightarrow y = e^{-2x} + \mathrm{C}e^{-3x}$

Hence, the required general solution of the given differential equation is

$y = e^{-2x} + \mathrm{C}e^{-3x}$ **[1 Mark]**

3. $\frac{dy}{dx} + \frac{y}{x} = x^2$ **[2 Marks]**

Solution:

The given differential equation is $\frac{dy}{dx} + \frac{y}{x} = x^2$

This is in the form of $\frac{dy}{dx} + Py = Q\left(\text{where } P = \frac{1}{x} \text{and } Q = x^2\right)$

Now, $I.F = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$ **[1 Mark]**

The solution of the given differential equation is given by the relation,

$y(\text{I.F.}) = \int (\text{Q} \times \text{I.F.})\, dx + \text{C}$

$\Rightarrow y(x) = \int (x^2 \cdot x)\, dx + \text{C}$

$\Rightarrow xy = \int x^3\, dx + \text{C}$

$\Rightarrow xy = \frac{x^4}{4} + \text{C}$

Hence, the required general solution of the given differential equation is $xy = \frac{x^4}{4} + \text{C}$ **[1 Mark]**

4. $\frac{dy}{dx} + \sec xy = \tan x \left(0 \le x < \frac{\pi}{2}\right)$ **[2 Marks]**

Solution:

The given differential equation is $\frac{dy}{dx} + \sec xy = \tan x \left(0 \le x < \frac{\pi}{2}\right)$

This is in the form of $\frac{dy}{dx}+Py=Q$ (where $P=\sec x$ and $Q=\tan x$)

Now $I.F=e^{\int Pdx}=e^{\int \sec x dx}=e^{\log(\sec x+\tan x)}=\sec x+\tan x$ **[1 Mark]**

The general solution of the given differential equation is given by the relation,

$y(\text{IF})=\int(\text{Q}\times\text{I.F.})\,dx+\text{C}$

$\Rightarrow y(\sec x+\tan x)=\int \tan x(\sec x+\tan x)dx+\text{C}$

$\Rightarrow y(\sec x+\tan x)=\int \sec x\tan x dx+\int \tan^2 x dx+\text{C}$

$\Rightarrow y(\sec x+\tan x)=\sec x+\int(\sec^2 x-1)\,dx+\text{C}$

$\Rightarrow y(\sec x+\tan x)=\sec x+\tan x-x+\text{C}$

Hence, the required general solution of the given differential equation is $y(\sec x+\tan x)=\sec x+\tan x-x+\text{C}$

[1 Mark]

5. $\cos^2 x\,.\frac{dy}{dx}+y=\tan x\ \left(0\le x<\frac{\pi}{2}\right)$

[4 Marks]

Solution:

Given differential equation is $\cos^2 x\,.\frac{dy}{dx}+y=\tan x$

$\Rightarrow \frac{dy}{dx}+\frac{y}{\cos^2 x}=\frac{\tan x}{\cos^2 x}$

$\Rightarrow \frac{dy}{dx}+y\sec^2 x=\tan x\sec^2 x$

This is in the form of $\frac{dy}{dx}+Py=Q$ (where $P=\sec^2 x$ and $Q=\tan x\sec^2 x$)

Now, $\text{I.F}=\text{e}^{\int pdx}$

$\text{I.F}=\text{e}^{\int \sec^2 x\cdot dx}$

$\text{I.F.}=\text{e}^{\tan x}$ **[1 Mark]**

Solution of the given equation is $y\times\text{I.F.}=\int Q\times\text{I.F}\times dx+C$

$y\cdot e^{\tan x}=\int \sec^2 x\cdot\tan x\cdot e^{\tan x}\cdot dx+C$ ..... (1)

Let $I = \int \sec^2 x \cdot \tan x \cdot e^{\tan x} \cdot dx$

Putting $t = \tan x$

$\Rightarrow \sec^2 x \cdot dx = dt$ **[1 Mark]**

$\mathrm{I} = \int \tan x \cdot \mathrm{e}^{\tan x} \cdot (\sec^2 x \cdot dx)$

$\Rightarrow \mathrm{I} = \int t \cdot e^t \cdot dt$

$\Rightarrow \mathrm{I} = \mathrm{t}\int e^t dt - \int \left[\frac{dt}{dt}\int e^t dt\right] dt$

$\Rightarrow \mathrm{I} = \mathrm{t} \cdot \mathrm{e}^{\mathrm{t}} - \int \mathrm{e}^{\mathrm{t}} dt$

$\Rightarrow \mathrm{I} = t\mathrm{e}^{\mathrm{t}} - \mathrm{e}^{\mathrm{t}}$ **[1 Mark]**

Putting $t = \tan x$, we get

$I = \tan x \cdot e^{\tan x} - e^{\tan x}$

$\Rightarrow I = e^{\tan x}(\tan x - 1)$

Substituting value of I in (1), we get

$ye^{\tan x} = e^{\tan x}(\tan x - 1) + C$

$\Rightarrow y = (\tan x - 1) + C.e^{-\tan x}$ **[1 Mark]**

Hence, the required general solution of the given differential equation is $y = (\tan x - 1) + C.e^{-\tan x}$

6. $x\frac{dy}{dx} + 2y = x^2 \log x$ **[4 Marks]**

Solution:

The given differential equation is $x\frac{dy}{dx} + 2y = x^2 \log x$

$$\Rightarrow \frac{dy}{dx} + \frac{2}{x}y = x\log x$$

This equation is in the form of a linear differential equation as:

$$\frac{dy}{dx} + py = Q\left(\text{where } p = \frac{2}{x} \text{ and } x\log x\right)$$

Now, $I.F = e^{\int p dx} = e^{\int \frac{2}{x} dx} = e^{2\log x} = e^{\log x^2} = x^2$ **[1 Mark]**

The general solution of the given differential equation is given by the relation,

$y(\text{I.F.}) = \int (\text{Q} \times \text{I.F.})\, dx + \text{C}$

$\Rightarrow y \cdot x^2 = \int (x\log x \cdot x^2)\, dx + \text{C}$

$\Rightarrow x^2 y = \int (x^3 \log x)\, dx + \text{C}$

$\Rightarrow x^2 y = \log x \cdot \int x^3\, dx - \int \left[\frac{d}{dx}(\log x) \cdot \int x^3\, dx\right] dx + \text{C}$ **[1 Mark]**

$\Rightarrow x^2 y = \log x \cdot \frac{x^4}{4} - \int \left(\frac{1}{x} \cdot \frac{x^4}{4}\right) dx + \text{C}$

$\Rightarrow x^2 y = \frac{x^4 \log x}{4} - \frac{1}{4} \int x^3\, dx + \text{C}$

$\Rightarrow x^2 y = \frac{x^4 \log x}{4} - \frac{1}{4} \cdot \frac{x^4}{4} + \text{C}$ **[1 Mark]**

$\Rightarrow x^2 y = \frac{1}{16} x^4 (4\log x - 1) + \text{C}$

$\Rightarrow y = \frac{1}{16} x^2 (4\log x - 1) + \text{C}x^{-2}$

Hence, the required general solution of the given differential equation is

$y = \frac{1}{16} x^2 (4\log x - 1) + \text{C}x^{-2}$ **[1 Mark]**

7. $x\log x \frac{dy}{dx} + y = \frac{2}{x} \log x$ **[4 Marks]**

Solution:

The given differential equation is $x\log x \frac{dy}{dx} + y = \frac{2}{x} \log x$

$\Rightarrow \frac{dy}{dx} + \frac{y}{x\log x} = \frac{2}{x^2}$

This equation is the form of a linear differential equation as:

$\frac{dy}{dx} + py = Q \left(\text{where } p = \frac{1}{x\log x} \text{ and } Q = \frac{2}{x^2}\right)$

Now, $I.F. = e^{\int p dx} = e^{\int \frac{1}{x\log} dx} = e^{\log(\log x)} = \log x$ **[1 Mark]**

The general solution of the given differential equation is given by the relation,

$y(I.F.) = \int (Q \times I.F.)dx + C$

$\Rightarrow y\log x = \int \left(\frac{2}{x^2}\log x\right)dx + C \ldots (1)$

Now, $\int \left(\frac{2}{x^2}\log x\right)dx = 2\int \left(\log x \cdot \frac{1}{x^2}\right)dx$

$= 2\left[\log x \cdot \int \frac{1}{x^2}dx - \int \left\{\frac{d}{dx}(\log x) \cdot \int \frac{1}{x^2}dx\right\}dx\right]$ **[1 Mark]**

$= 2\left[\log x\left(-\frac{1}{x}\right) - \int \left(\frac{1}{x} \cdot \left(-\frac{1}{x}\right)\right)dx\right]$

$= 2\left[-\frac{\log x}{x} + \int \frac{1}{x^2}dx\right]$

$= 2\left[-\frac{\log x}{x} - \frac{1}{x}\right]$

$= -\frac{2}{x}(1 + \log x)$ **[1 Mark]**

Substituting the value of $\int \left(\frac{2}{x^2}\log x\right)dx$ in equation (1), we get:

$y\log x = -\frac{2}{x}(1 + \log x) + C$

Hence, the required general solution of the given differential equation is

$y\log x = -\frac{2}{x}(1 + \log x) + C$ **[1 Mark]**

8. $(1 + x^2)dy + 2xydx = \cot x dx (x \neq 0)$ **[4 Marks]**

Solution:

Given differential equation is $(1 + x^2)dy + 2xydx = \cot x dx$

$\Rightarrow \frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{\cot x}{1+x^2}$ **[1 Mark]**

This equation is a linear differential equation of the form:

$\frac{dy}{dx} + py = Q \left(\text{where } p = \frac{2x}{1+x^2} \text{ and } Q = \frac{\cot x}{1+x^2}\right)$

Now, $I.F = e^{\int p dx} = e^{\int \frac{2x}{1+x^2} dx} = e^{\log(1+x^2)} = 1 + x^2$ **[1 Mark]**

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F.}) = \int (\text{Q} \times \text{I.F.})\, dx + C$$

$$\Rightarrow y(1+x^2) = \int \left[\frac{\cot x}{1+x^2} \times (1+x^2)\right] dx + C$$

$$\Rightarrow y(1+x^2) = \int \cot x dx + C$$

$$\Rightarrow y(1+x^2) = \log|\sin x| + C$$

Hence, the required general solution of the given differential equation is

$y(1+x^2) = \log|\sin x| + C$ **[2 Marks]**

9. $x\frac{dy}{dx} + y - x + xy\cot x = 0 (x \neq 0)$ **[4 Marks]**

Solution:

Given differential equation is $x\frac{dy}{dx} + y - x + xy\cot x = 0$

$$\Rightarrow x\frac{dy}{dx} + y(1 + x\cot x) = x$$

$\Rightarrow \frac{dy}{dx} + \left(\frac{1}{x} + \cot x\right) y = 1$ **[1 Mark]**

This equation is a linear differential equation of the form:

$$\frac{dy}{dx} + py = Q \left(\text{where } p = \frac{1}{x} + \cot x \text{ and } Q = \frac{\cot x}{1+x^2}\right)$$

Now, $I.F = e^{\int p dx} = e^{\int \frac{2x}{1+x^2} dx} = e^{\int \log(1+x^2)} = 1 + x^2$ **[1 Mark]**

The general solution of the given differential equation is given by the relation,

$$y(I.F.) = \int (\text{Q} \times \text{I.F.})\, dx + \text{C}$$

$$\Rightarrow y(1+x^2) = \int \left[\frac{\cot x}{1+x^2} \times (1+x^2)\right] dx + \text{C}$$

$\Rightarrow y(1+x^2) = \int \cot x dx + C$

$\Rightarrow y(1+x^2) = \log|\sin x| + C$

Hence, the required general solution of the given differential equation is

$y(1+x^2) = \log|\sin x| + C$ **[2 Marks]**

10. $x\frac{dy}{dx} + y - x + xy\cot x = 0 (x \neq 0)$ **[4 Marks]**

Solution:

Given differential equation is $x\frac{dy}{dx} + y - x + xy\cot x = 0$

$$\Rightarrow x\frac{dy}{dx} + y(1 + x\cot x) = x$$

$\Rightarrow \frac{dy}{dx} + \left(\frac{1}{x} + \cot x\right)y = 1$ **[1 Mark]**

The equation is a linear differential equation of the form:

$\frac{dy}{dx} + py = Q$ $\left(\text{where } p = \frac{1}{x}\cot x \text{ and } Q = 1\right)$

Now, $I.F = e^{\int p dx} = e^{\int\left(\frac{1}{x}+\cot x\right)dx} = e^{\log x + \log(\sin x)} = e^{\int \log(x \sin x)} = x \sin x$ **[1 Mark]**

The general solution of the given differential equation is given by the relation,

$$y(I.F.) = \int (Q \times I.F.)\, dx + C$$

$$\Rightarrow y(x\sin x) = \int (1 \times x\sin x)\, dx + C$$

$$\Rightarrow y(x\sin x) = \int (x\sin x)\, dx + C$$

$$\Rightarrow y(x\sin x) = x\int \sin x dx - \int \left[\frac{d}{dx}(x) \cdot \int \sin x\, dx\right] + C$$

$$\Rightarrow y(x\sin x) = x(-\cos x) - \int 1 \cdot (-\cos x) dx + C$$

$$\Rightarrow y(x\sin x) = -x\cos x + \sin x + C$$

$$\Rightarrow y = \frac{-x\cos x}{x\sin x} + \frac{\sin x}{x\sin x} + \frac{C}{x\sin x}$$

$\Rightarrow y = -\cot \cdot x + \frac{1}{x} + \frac{C}{x\sin x}$ **[2 Marks]**

Hence, the required general solution of the given differential equation is $y = -\cot \cdot x + \frac{1}{x} + \frac{C}{x\sin x}$

11. $(x + y)\frac{dy}{dx} = 1$ **[4 Marks]**

Solution:

Given differential equation is $(x + y)\frac{dy}{dx} = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + y}$$

$$\Rightarrow \frac{dx}{dy} = x + y$$

$$\Rightarrow \frac{dx}{dy} - x = y$$

This is a linear differential equation of the form:

$\frac{dy}{dx} + px = Q$ (where $p = -1$ and $Q = y$)

Now, $I.F = e^{\int \rho dy} = e^{\int -dy} = e^{-y}$. **[2 Marks]**

The general solution of the given differential equation is given by the relation,

$$x(I.F.) = \int (Q \times \text{I.F.})\, dy + C$$

$$\Rightarrow xe^{-y} = \int (y \cdot e^{-y})\, dy + C$$

$$\Rightarrow xe^{-y} = y \cdot \int e^{-y}\, dy - \int \left[\frac{d}{dy}(y) \int e^{-y}\, dy\right] dy + C$$

$$\Rightarrow xe^{-y} = y(-e^{-y}) - \int (-e^{-y})\, dy + C$$

$$\Rightarrow xe^{-y} = -ye^{-y} + \int e^{-y}\, dy + C$$

$$\Rightarrow xe^{-y} = -ye^{-y} - e^{-y} + C$$

$$\Rightarrow x = -y - 1 + Ce^{y}$$

$\Rightarrow x + y + 1 = \text{C}e^{y}$ **[2 Marks]**

Hence, the required general solution of the given differential equation is $x + y + 1 = \text{C}e^{y}$

12. $ydx + (x - y^2)dy = 0$ **[4 Marks]**

Solution:

Given differential equation is $ydx + (x - y^2)dy = 0$

$\Rightarrow ydx = (y^2 - x)dy$

$\Rightarrow \frac{dx}{dy} = \frac{y^2 - x}{y} = y - \frac{x}{y}$

$\Rightarrow \frac{dx}{dy} + \frac{x}{y} = y$ **[1 Mark]**

This is a linear differential equation of the form:

$\frac{dy}{dx} + px = Q \left(\text{where } p = \frac{1}{y} \text{ and } Q = y\right)$

Now $I.F = e^{\int \rho dy} = e^{\int \frac{1}{y} dy} = e^{\log y} = y.$ **[1 Mark]**

The general solution of the given differential equation is given by the relation,

$x(\text{I.F.}) = \int (\text{Q} \times \text{I.F.})\, dy + \text{C}$

$\Rightarrow xy = \int (y \cdot y) dy + \text{C}$

$\Rightarrow xy = \int y^2 dy + \text{C}$

$\Rightarrow xy = \frac{y^3}{3} + \text{C}$

$\Rightarrow x = \frac{y^2}{3} + \frac{\text{C}}{y}$

Hence, the required general solution of the given differential equation is $x = \frac{y^2}{3} + \frac{\text{C}}{y}$ **[2 Marks]**

13. $(x+3y^2)\frac{dy}{dx} = y(y > 0)$ **[4 Marks]**

Solution:

Given differential equation is $(x+3y^2)\frac{dy}{dx} = y$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x+3y^2}$$

$$\Rightarrow \frac{dx}{dy} = \frac{x+3y^2}{y} = \frac{x}{y} + 3y$$

$\Rightarrow \frac{dx}{dy} - \frac{x}{y} = 3y$ **[1 Mark]**

This is a linear differential equation of the form:

$$\frac{dx}{dy} + px = Q \left(\text{where } p = -\frac{1}{y} \text{and } Q = 3y\right)$$

Now, $I.F = e^{\int p dy} = e^{-\int \frac{dv}{y}} = e^{-\log y} = e^{\log\left(\frac{1}{y}\right)} = \frac{1}{y}.$ **[1 Mark]**

The general solution of the given differential equation is given by the relation,

$$x(\text{I.F.}) = \int (\text{Q} \times \text{IF})\, dy + \text{C}$$

$$\Rightarrow x \times \frac{1}{y} = \int \left(3y \times \frac{1}{y}\right) dy + \text{C}$$

$$\Rightarrow \frac{x}{y} = 3y + \text{C}$$

$$\Rightarrow x = 3y^2 + \text{Cy}$$

Hence, the required general solution of the given differential equation is $x = 3y^2 + \text{Cy}$ **[2 Marks]**

14. $\frac{dy}{dx} + 2y\tan x = \sin x; y = 0 \text{ when } x = \frac{\pi}{3}$ **[4 Marks]**

Solution:

The given differential equation is $\frac{dy}{dx} + 2y\tan x = \sin x$

This is a linear equation of the form:

$\frac{dy}{dx} + py = Q$(where $p = 2\tan x$ and $Q = \sin x$

Now, $I.F = e^{\int pdx} = e^{\int 2\tan x dx} = e^{2\log|\sec x|} = e^{\log(\sec^2 x)} = \sec^2 x.$ **[1 Mark]**

The general solution of the given differential equation is given by the relation,

$y(\text{I.F.}) = \int (\text{Q} \times \text{I.F,})\, dx + \text{C}$

$\Rightarrow y(\sec^2 x) = \int (\sin x \cdot \sec^2 x)\, dx + \text{C}$

$\Rightarrow y\sec^2 x = \int (\sec x \cdot \tan x)\, dx + \text{C}$

$\Rightarrow y\sec^2 x = \sec x + \text{C} \ldots (1)$ **[1 Mark]**

Now, $y = 0$ at $x = \frac{\pi}{3}$

Therefore,

$0 \times \sec^2 \frac{\pi}{3} = \sec\frac{\pi}{3} + \text{C}$

$\Rightarrow 0 = 2 + \text{C}$

$\Rightarrow \text{C} = -2$ **[1 Mark]**

Substituting $C =- 2$ in equation (1), we get:

$y\sec^2 x = \sec x - 2$

$\Rightarrow y = \cos x - 2\cos^2 x$

Hence, the required solution of the given differential equation is $y = \cos x - 2\cos^2 x$. **[1 Mark]**

15. $(1 + x^2)\frac{dy}{dx} + 2xy = \frac{1}{1+x^2}; y = 0$ when $x = 1$ **[4 Marks]**

Solution:

Given differential equation is $(1 + x^2)\frac{dy}{dx} + 2xy = \frac{1}{1+x^2}$

$\Rightarrow \frac{dy}{dx} + \frac{2xy}{1 + x^2} = \frac{1}{(1 + x^2)^2}$

This is a linear differential equation of the form:

$$\frac{dy}{dx}+py=Q\left(\text{where } p=\frac{2x}{1+x^2} \text{ and } Q=\frac{1}{(1+x^2)^2}\right)$$

Now, $I.F=e^{\int \rho dx}=e^{\int\frac{2xdx}{1+x^2}}=e^{\log(1+x^2)}=1+x^2$ **[1 Mark]**

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F})=\int(\text{Q}\times\text{I.F})\,dx+\text{C}$$

$$\Rightarrow y(1+x^2)=\int\left[\frac{1}{(1+x^2)^2}\cdot(1+x^2)\right]dx+\text{C}$$

$$\Rightarrow y(1+x^2)=\int\frac{1}{1+x^2}dx+C$$

$\Rightarrow y(1+x^2)=\tan^{-1}x+C\ ...(1)$ **[1 Mark]**

Now, $y=0$ at $x=1$.

Therefore,

$0=\tan^{-1}1+\text{C}$

$\Rightarrow \text{C}=-\frac{\pi}{4}$ **[1 Mark]**

Substituting $C=-\frac{\pi}{4}$ in equation (1), we get:

$$y(1+x^2)=\tan^{-1}x-\frac{\pi}{4}$$

Hence, the required general solution of the given differential equation is

$y(1+x^2)=\tan^{-1}x-\frac{\pi}{4}$ **[1 Mark]**

16. $\frac{dy}{dx}-3y\cot x=\sin 2x; y=2$ when $x=\frac{\pi}{2}$ **[4 Marks]**

Solution:

The given differential equation is $\frac{dy}{dx}-3y\cot x=\sin 2x$

This is a linear differential equation of the form:

$\frac{dy}{dx} + py = Q$ (where $p = -3\cot x$ and $Q = \sin x$)

Now, $I.F = e^{\int p dx} = e^{-3\int \cot x dx} = e^{-3\log|\sin x|} = e^{\log\left|\frac{1}{|\sin^3 x|}\right|} = \frac{1}{\sin^3 x}$ **[1 Mark]**

The general solution of the given differential equation is given by the relation,

$y(\text{I.F.}) = \int (\text{Q} \times \text{I.F.})dx + \text{C}$

$\Rightarrow y \cdot \frac{1}{\sin^3 x} = \int \left[\sin 2x \cdot \frac{1}{\sin^3 x}\right] dx + \text{C}$ **[1 Mark]**

$\Rightarrow y\operatorname{cosec}^3 x = 2\int (\cot x \operatorname{cosec} x)\, dx + \text{C}$

$\Rightarrow y\operatorname{cosec}^3 x = 2\operatorname{cosec} x + C$

$\Rightarrow y = -\frac{2}{\operatorname{cosec}^2 x} + \frac{3}{\operatorname{cosec}^3 x}$

$\Rightarrow y = -2\sin^2 x + \text{C}\sin^3 x \ldots (1)$

Now, $y = 2$ at $x = \frac{\pi}{2}$

Therefore, we get:

$2 = -2 + \text{C}$

$\Rightarrow \text{C} = 4$ **[1 Mark]**

Substituting $C = 4$ in equation (1), we get:

$y = -2\sin^2 x + 4\sin^3 x$

$\Rightarrow y = 4\sin^3 x - 2\sin^2 x$

Hence, the required particular solution of the given differential equation is

$y = 4\sin^3 x - 2\sin^2 x$ **[1 Mark]**

17. Find the equation of a curve passing through the origin given that the slope of the tangent to the curve at any point (x, y) is equal to the sum of the coordinates of the point. **[4 Marks]**

Solution:

Let $F(x, y)$ be the curve passing through the origin.

At point (x, y), the slope of the curve will be $\frac{dy}{dx}$

According to the given information:

$$\frac{dy}{dx} = x + y$$

$$\Rightarrow \frac{dy}{dx} - y = x$$

This is a linear differential equation of the form:

$\frac{dy}{dx} + py = Q$ (where $p = -1$ and $Q = x$)

Now, $I.F = e^{\int p dx} = e^{\int(-1)dx} = e^{-x}$. **[1 Mark]**

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F.}) = \int (\text{Q} \times \text{I.F.})\, dx + \text{C}$$

$$\Rightarrow ye^{-x} = \int x\, e^{-x} dx + \text{C} \ldots (1)$$

Now, $\int xe^{-x}\, dx = x \int e^{-x}\, dx - \int \left[\frac{d}{dx}(x) \cdot \int e^{-x}\, dx\right] dx$

$= -xe^{-x} - \int -e^{-x}\, dx$ **[1 Mark]**

$= -xe^{-x} + (-e^{-x})$

$= -e^{-x}(x + 1)$

Substituting in equation (1), we get:

$ye^{-x} = -e^{-x}(x + 1) + C$

$\Rightarrow y = -(x + 1) + Ce^{x}$

$\Rightarrow x + y + 1 = Ce^{x} \ldots (2)$ **[1 Mark]**

The curve passes through the origin.

Therefore, equation (2) becomes:

$1 = C$

$\Rightarrow C = 1$

Substituting $C = 1$ in equation (2), we get:

$x + y + 1 = e^{x}$

Hence, the required equation of curve passing through the origin is $x + y + 1 = e^{x}$ **[1 Mark]**

18. Find the equation of a curve passing through the point $(0,2)$ given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5.

[4 Marks]

Solution:

Let $F(x,y)$ be the curve and let (x,y) be a point on the curve. The slope of the tangent to the curve at (x,y) is $\frac{dy}{dx}$.

According to the given information:

$$\frac{dy}{dx}+5=x+y$$

$$\Rightarrow \frac{dy}{dx}-y=x-5$$

This is a linear differential equation of the form:

$$\frac{dy}{dx}+py=Q(\text{where } p=-1 \text{ and } Q=x-5)$$

Now, $I.F=e^{\int pkx}=e^{\int(-1)dx}=e^{-x}$. **[1 Mark]**

The general equation of the curve is given by the relation,

$$y(\text{I.F.})=\int(\text{Q}\times\text{IF})\,dx+\text{C}$$

$$\Rightarrow y\cdot e^{-x}=\int(x-5)\,e^{-x}dx+\text{C}$$

Now, $\int(x-5)\,e^{-x}dx=(x-5)\int e^{-x}\,dx-\int\left[\frac{d}{dx}(x-5)\cdot\int e^{-x}\,dx\right]dx$

$=(x-5)(-e^{-x})-\int(-e^{-x})\,dx$ **[1 Mark]**

$=(5-x)e^{-x}+(-e^{-x})$

$=(4-x)e^{-x}$

Therefore, equation (1) becomes:

$$ye^{-x}=(4-x)e^{-x}+C$$

$$\Rightarrow y=4-x+Ce^{x}$$

$\Rightarrow x+y-4=Ce^{x}\ \dots(2)$ **[1 Mark]**

The curve passes through point $(0,2)$.

Therefore, equation (2) becomes:

$0 + 2 - 4 = Ce^0$

$\Rightarrow -2 = C$

$\Rightarrow C = -2$

Substituting $C = -2$ in equation (2), we get:

$x + y - 4 = -2e^x$

$\Rightarrow y = 4 - x - 2e^x$

Hence, the required equation of the curve is $y = 4 - x - 2e^x$ **[1 Mark]**

19. The integrating factor of the differential equation $x\frac{dy}{dx} - y = 2x^2$ is **[2 Marks]**

(A) e^{-x}

(B) e^{-y}

(C) $\frac{1}{x}$

(D) x

Solution:

The given differential equation is $x\frac{dy}{dx} - y = 2x^2$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = 2x$$

This is a linear differential equation of the form:

$\frac{dy}{dx} + py = Q$ $\left(\text{where } p = -\frac{1}{x} \text{and } Q = 2x\right)$ **[1 Mark]**

The integrating factor $(I.F)$ is given by the relation,

$e^{\int p dx}$

$$\therefore I.F = e^{\int \frac{1}{x} dx} = e^{-\log x} = e^{\log(x^{-1})} = x^{-1} = \frac{1}{x}$$

Hence, the correct answer is C. **[1 Mark]**

20. The integrating factor of the differential equation.

$(1-y^2)\frac{dx}{dy}+yx=ay(-1<y<1)$ **[2 Marks]**

(A) $\frac{1}{y^2-1}$

(B) $\frac{1}{\sqrt{y^2-1}}$

(C) $\frac{1}{1-y^2}$

(D) $\frac{1}{\sqrt{1-y^2}}$

Solution:

The given differential equation is $(1-y^2)\frac{dx}{dy}+yx=ay$

$$\Rightarrow \frac{dy}{dx}+\frac{yx}{1-y^2}=\frac{ay}{1-y^2}$$

This is a linear differential equation of the form:

$\frac{dx}{dy}+py=Q\left(\text{where } p=\frac{y}{1-y^2} \text{ and } Q=\frac{ay}{1-y^2}\right)$ **[1 Mark]**

The integrating factor (I.F) is given by the relation,

$e^{\int pdt}$

$$\therefore \text{I.F}=e^{\int pdy}=e^{\int\frac{y}{1-y^2}dy}=e^{-\frac{1}{2}\log(1-y^2)}=e^{\log\left[\frac{1}{\sqrt{1-y^2}}\right]}=\frac{1}{\sqrt{1-y^2}}$$

Hence, the correct answer is D. **[1 Mark]**

Miscellaneous exercise 9

1. For each of the differential equations given below, indicate its order and degree (if defined).

(I) $\frac{d^2y}{dx^2}+5x\left(\frac{dy}{dx}\right)^2-6y=\log x$ **[1 Mark]**

(II) $\left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)^2+7y=\sin x$ **[1 Mark]**

(III) $\frac{d^4y}{dx^4} - \sin\left(\frac{d^3y}{dx^3}\right) = 0$ **[1 Mark]**

Solution:

(I)The differential equation is given as $\frac{d^2y}{dx^2} + 5x\left(\frac{dy}{dx}\right)^2 - 6y = \log x$

$$\Rightarrow \frac{d^2y}{dx^2} + 5x\left(\frac{dy}{dx}\right)^2 - 6y - \log x = 0$$

The highest order derivative present in the differential equation is $\frac{d^2y}{dx^2}$. Thus, its order is two. The highest power raised to $\frac{d^2y}{dx^2}$ is one. Hence, its degree is one.

[1 Mark]

(II) The differential equation is given as:

$$\left(\frac{dy}{dx}\right)^3 - 4\left(\frac{dy}{dx}\right)^2 + 7y = \sin x$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^3 - 4\left(\frac{dy}{dx}\right)^2 + 7y - \sin x = 0$$

The highest order derivative present in the differential equation is $\frac{dy}{dx}$. Thus, its order is one. The highest power raised to is three. Hence, its degree is three.

[1 Mark]

(iii) The differential equation is given as:

$$\frac{d^4y}{dx^4} - \sin\left(\frac{d^3y}{dx^3}\right) = 0$$

The highest order derivative present in the differential equation is $\frac{d^4y}{dx^4}$. Thus, its order is four.

However, the given differential equation is not a polynomial equation.

Hence, its degree is not defined. **[1 Mark]**

2. For each of the exercises given below, verify that the given function (implicit or explicit) is a solution of the corresponding differential equation.

(I) $y = ae^x + be^{-x} + x^2 : x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0$ **[2 MarkS]**

(II) $y = e^x(a\cos x + b\sin x): \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$ **[4 MarkS]**

(III) $y = x\sin 3x: \frac{d^2y}{dx^2} + 9y - 6\cos 3x = 0$ **[2 MarkS]**

(IV) $x^2 = 2y^2\log y: (x^2 + y^2)\frac{dy}{dx} - xy = 0$ **[2 MarkS]**

Solution:

(I) $y = ae^x + be^{-x} + x^2$

Differentiating both sides with respect to x, we get:

$$\frac{dy}{dx} = a\frac{d}{dx}(e^x) + b\frac{d}{dx}(e^{-x}) + \frac{d}{dx}(x^2)$$

$$\Rightarrow \frac{dy}{dx} = ae^x - be^{-x} + 2x$$

Again, differentiating both sides with respect to x, we get:

$\frac{d^2y}{dx^2} = ae^x + be^{-x} + 2$ **[1 Mark]**

Now, on substituting the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the differential equation, we get:

L.H.S.

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2$$

$$= x(ae^x + be^{-x} + 2) + 2(ae^x - be^{-x} + 2x) - x(ae^x + be^{-x} + x^2) + x^2 - 2$$

$$= (axe^x + bxe^{-x} + 2x) + (2ae^x - 2be^{-x} + 4x) - (axe^x + bxe^{-x} + x^3) + x^2 - 2$$

$$= 2ae^x - 2be^{-x} + x^2 + 6x - 2$$

$$\neq 0$$

$\Rightarrow \text{L.H.S.} \neq \text{R.H.S.}$ **[1 Mark]**

Hence, the given function is not a solution of the corresponding differential equation.

(II) $y = e^x(a\cos x + b\sin x) = ae^x\cos x + be^x\sin x$

Differentiating both sides with respect to x, we get:

$$\frac{dy}{dx} = a\cdot\frac{d}{dx}(e^x\cos x) + b\cdot\frac{d}{dx}(e^x\sin x)$$

$\Rightarrow \dfrac{dy}{dx} = a(e^x\cos x - e^x\sin x) + b\cdot(e^x\sin x + e^x\cos x)$

$\Rightarrow \frac{dy}{dx} = (a+b)e^x\cos x + (b-a)e^x\sin x$ **[1 Mark]**

Again, differentiating both sides with respect to x, we get:

$\frac{d^2y}{dx^2} = (a+b)\cdot\frac{d}{dx}(e^x\cos x) + (b-a)\frac{d}{dx}(e^x\sin x)$

$\Rightarrow \dfrac{d^2y}{dx^2} = (a+b)\cdot[e^x\cos x - e^x\sin x] + (b-a)[e^x\sin x + e^x\cos x]$

$\Rightarrow \dfrac{d^2y}{dx^2} = e^x[(a+b)(\cos x - \sin x) + (b-a)(\sin x + \cos x)]$

$\Rightarrow \dfrac{d^2y}{dx^2} = e^x[a\cos x - a\sin x + b\cos x - b\sin x + b\sin x + b\cos x - a\sin x - a\cos x]$

$\Rightarrow \frac{d^2y}{dx^2} = [2e^x(b\cos x - a\sin x)]$ **[1 Mark]**

Now, on substituting the values of $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ in the L.H.S. of the given differential equation, we get:

$\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y$

$= 2e^x(b\cos x - a\sin x) - 2e^x[(a+b)\cos x + (b-a)\sin x] + 2e^x(a\cos x + b\sin x)$

$= e^x\begin{bmatrix}(2b\cos x - 2a\sin x) - (2a\cos x + 2b\cos x) \\ -(2b\sin x - 2a\sin x) + (2a\cos x + 2b\sin x)\end{bmatrix}$

$= e^x[(2b - 2a - 2b + 2a)\cos x] + e^x[(-2a - 2b + 2a + 2b)\sin x]$

$= 0$

Hence, the given function is a solution of the corresponding differential equation. **[2 MarkS]**

(III) $y = x\sin 3x$

Differentiating both sides with respect to x, we get:

$\dfrac{dy}{dx} = \dfrac{d}{dx}(x\sin 3x) = \sin 3x + x\cdot\cos 3x\cdot 3$

$\Rightarrow \dfrac{dy}{dx} = \sin 3x + 3x\cos 3x$

Again, differentiating both sides with respect to x, we get:

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(\sin 3x) + 3\dfrac{d}{dx}(x\cos 3x)$

$$\Rightarrow \frac{d^2y}{dx^2} = 3\cos 3x + 3[\cos 3x + x(-\sin 3x)\cdot 3]$$

$\Rightarrow \frac{d^2y}{dx^2} = 6\cos 3x - 9x\sin 3x$ **[1 Mark]**

Substituting the value of $\frac{d^2y}{dx^2}$ in the L.H.S. of the given differential equation, we get:

$$\frac{d^2y}{dx^2} + 9y - 6\cos 3x$$

$$= (6\cdot\cos 3x - 9x\sin 3x) + 9x\sin 3x - 6\cos 3x$$

$$= 0$$

Hence, the given function is a solution of the corresponding differential equation. **[1 Mark]**

(IV) $x^2 = 2y^2 \log y$

Differentiating both sides with respect to x, we get:

$$2x = 2\cdot\frac{d}{dx} = [y^2\log y]$$

$$\Rightarrow x = \left[2y\cdot\log y\cdot\frac{dy}{dx} + y^2\cdot\frac{1}{y}\cdot\frac{dy}{dx}\right]$$

$$\Rightarrow x = \frac{dy}{dx}(2y\log y + y)$$

$\Rightarrow \frac{dy}{dx} = \frac{x}{y(1+2\log y)}$ **[1 Mark]**

Substituting the value of $\frac{dy}{dx}$ in the L.H.S. of the given differential equation, we get:

$$(x^2+y^2)\frac{dy}{dx} - xy$$

$$= (2y^2\log y + y^2)\cdot\frac{x}{y(1+2\log y)} - xy$$

$$= y^2(1+2\log y)\cdot\frac{x}{y(1+2\log y)} - xy$$

$$= xy - xy$$

$$= 0$$

Hence, the given function is a solution of the corresponding differential equation. **[1 Mark]**

3. Form the differential equation representing the family of curves given by

$(x-a)^2 + 2y^2 = a^2$

where a is an arbitrary constant. **[2 Marks]**

Solution:

$(x-a)^2 + 2y^2 = a^2$

$\Rightarrow x^2 + a^2 - 2ax + 2y^2 = a^2$

$\Rightarrow 2y^2 = 2ax - x^2 \ldots (1)$

Differentiating with respect to x, we get:

$$2y\frac{dy}{dx} = \frac{2a-2x}{2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{a-x}{2y}$$

$\Rightarrow \frac{dy}{dx} = \frac{2ax-2x^2}{4xy} \ldots (2)$ **[1 Mark]**

From equation (1), we get:

$2ax = 2y^2 + x^2$

On substituting this value in equation (3), we get:

$$\frac{dy}{dx} = \frac{2y^2 + x^2 - 2x^2}{4xy}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2y^2 - x^2}{4xy}$$

Hence, the differential equation of the family of curves is given as $\frac{dy}{dx} = \frac{2y^2-x^2}{4xy}$ **[1 Mark]**

4. Prove that $x^2 - y^2 = c(x^2+y^2)^2$ is the general solution of differential equation, $(x^3 - 3xy^2)dx = (y^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$ where c is a parameter. **[6 Marks]**

Solution:

$(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$

$\Rightarrow \frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y} \dots (1)$

This is a homogeneous equation. To simplify it, we need to make the substitution as:

$y = vx$

$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ **[1 Mark]**

Substituting the values of y and $\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \frac{x^3 - 3x(vx)^2}{(vx)^3 - 3x^2(vx)}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v}$

$\Rightarrow x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v} - v$

$\Rightarrow x\frac{dv}{dx} = \frac{1 - 3v^2 - v(v^3 - 3v)}{v^3 - 3v}$

$\Rightarrow x\frac{dv}{dx} = \frac{1 - v^4}{v^3 - 3v}$

$\Rightarrow \left(\frac{v^3 - 3v}{1 - v^4}\right)dv = \frac{dx}{x}$ **[1 Mark]**

Integrating both sides, we get:

$\int \left(\frac{v^3 - 3v}{1 - v^4}\right)dv = \log x + \log C' \dots (2)$

Now, $\int \left(\frac{v^3 - 3v}{1 - v^4}\right)dv = \int \frac{v^3 dv}{1 - v^4} - 3\int \frac{v dv}{1 - v^4}$

$\Rightarrow \int \left(\frac{v^3 - 3v}{1 - v^4}\right)dv = I_1 - 3I_2$, where $I_1 = \int \frac{v^3 dv}{1 - v^4}$ and $I_2 = \int \frac{v dv}{1 - v^4} \dots (3)$ **[1 Mark]**

Let $1 - v^4 = t$

$\therefore \frac{d}{dv}(1 - v^4) = \frac{dt}{dv}$

$\Rightarrow -4v^3 = \frac{dt}{dv}$

$\Rightarrow v^3 dv = -\frac{dt}{4t} = -\frac{1}{4}\log t = -\frac{1}{4}\log(1 - v^4)$

And, $I_2 = \int \frac{v dv}{1 - v^4} = \int \frac{v dv}{1 - (v^2)^2}$ **[1 Mark]**

Let $v^2 = p$

$$\therefore \frac{d}{dv}(v^2) = \frac{dp}{dv}$$

$$\Rightarrow 2v = \frac{dp}{dv}$$

$$\Rightarrow vdv = \frac{dp}{2}$$

$$\Rightarrow I_2 = \frac{1}{2}\int \frac{dp}{1-p^2} = \frac{1}{2\times2}\log\left|\frac{1+p}{1-p}\right| = \frac{1}{4}\log\left|\frac{1+v^2}{1-v^2}\right|$$ **[1 Mark]**

Substituting the values of I_1 and I_2 in equation (3), we get:

$$\int\left(\frac{v^3-3v}{1-v^4}\right)dv = -\frac{1}{4}\log(1-v^4) - \frac{3}{4}\log\left|\frac{1-v^2}{1+v^2}\right|$$

Therefore, equation (2) becomes:

$$\frac{1}{4}\log(1-v^4) - \frac{3}{4}\log\left|\frac{1+v^2}{1-v^2}\right| = \log x + \log C'$$

$$\Rightarrow -\frac{1}{4}\log\left[(1-v^4)\left(\frac{1+v^2}{1-v^2}\right)^3\right] = \log C'x$$

$$\Rightarrow \frac{(1+v^2)^4}{(1-v^2)^2} = (C'x)^{-4}$$

$$\Rightarrow \frac{\left(1+\frac{y^2}{x^2}\right)^4}{\left(1-\frac{y^2}{x^2}\right)^2} = \frac{1}{C'^4x^4}$$

$$\Rightarrow \frac{(x^2+y^2)^4}{x^4(x^2-y^2)^2} = \frac{1}{C^4x^4}$$

$$\Rightarrow (x^2-y^2)^2 = C^4(x^2+y^2)^4$$

$\Rightarrow x^2 - y^2 = C(x^2+y^2)^2$, where $C = C'^2$

Hence proved. **[1 Mark]**

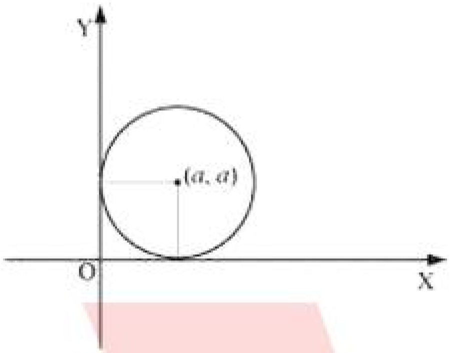
5. Form the differential equation of the family of circles in the first quadrant which touch the coordinate axes.

[4 Marks]

Solution:

The equation of a circle in the first quadrant with centre (a, a) and radius (a) which touches the coordinate axes is:

$(x-a)^2+(y-a)^2=a^2 \ldots(1)$



Differentiating equation (1) with respect to x, we get:

$2(x-a)+2(y-a)\frac{dy}{dx}=0$

$\Rightarrow (x-a)+(y-a)y'=0$

$\Rightarrow x-a+yy'-ay'=0$

$\Rightarrow x+yy'-a(1+y')=0$

$\Rightarrow a=\frac{x+yy'}{1+y'}$ **[2 Marks]**

Substituting the value of a in equation (1), we get:

$\left[x-\left(\frac{x+yy'}{1+y'}\right)\right]^2+\left[y-\left(\frac{x+yy'}{1+y'}\right)\right]^2=\left(\frac{x+yy'}{1+y'}\right)^2$

$\Rightarrow \left[\frac{(x-y)y'}{(1+y')}\right]^2+\left[\frac{y-x}{1+y'}\right]^2=\left[\frac{x+yy'}{1+y'}\right]^2$

$\Rightarrow (x-y)^2\cdot y'^2+(x-y)^2=(x+yy')^2$

$\Rightarrow (x-y)^2[1+(y')^2]=(x+yy')^2$

Hence, the required differential equation of the family of circles is

$(x-y)^2[1+(y')^2]=(x+yy')^2$. **[2 Marks]**

6. Find the general solution of the differential equation $\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$

[2 Marks]

Solution:

Given differential equation is $\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$

$\Rightarrow \frac{dy}{dx}=-\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

$\Rightarrow \frac{dy}{\sqrt{1-y^2}}=\frac{-dx}{\sqrt{1-x^2}}$ **[1 Mark]**

Integrating both sides, we get:

$\sin^{-1}y=-\sin^{-1}x+C$

$\Rightarrow \sin^{-1}x+\sin^{-1}y=C$

Hence, the general solution of given differential equation is $\sin^{-1}x+\sin^{-1}y=C$ **[1 Mark]**

7. Show that the general solution of the differential equation $\frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+1}=0$ is given by $(x+y+1)=A(1-x-y-2xy)$, where A is parameter **[4 Marks]**

Solution:

Given differential equation is $\frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+1}=0$

$\Rightarrow \frac{dy}{dx}=-\frac{(y^2+y+1)}{x^2+x+1}$

$\Rightarrow \frac{dy}{y^2+y+1}=\frac{-dx}{x^2+x+1}$

$\Rightarrow \frac{dy}{y^2+y+1}+\frac{dx}{x^2+x+1}=0$ **[1 Mark]**

Integrating both sides, we get:

$\int\frac{dy}{y^2+y+1}+\int\frac{dx}{x^2+x+1}=\text{C}$

$\Rightarrow \int\frac{dy}{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\int\frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\text{C}$

$\Rightarrow \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right]+\frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right]=C$ **[1 Mark]**

$$\Rightarrow \tan^{-1}\left[\frac{2y+1}{\sqrt{3}}\right]+\tan^{-1}\left[\frac{2x+1}{\sqrt{3}}\right]=\frac{\sqrt{3}C}{2}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{2y+1}{\sqrt{3}}+\frac{2x+1}{\sqrt{3}}}{1-\frac{(2y+1)}{\sqrt{3}}\cdot\frac{(2x+1)}{\sqrt{3}}}\right]=\frac{\sqrt{3}C}{2}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{2x+2y+2}{\sqrt{3}}}{1-\left(\frac{4xy+2x+2y+1}{3}\right)}\right]=\frac{\sqrt{3}C}{2}$$

$$\Rightarrow \tan^{-1}\left[\frac{2\sqrt{3}(x+y+1)}{3-4xy-2x-2y-1}\right]=\frac{\sqrt{3}C}{2}$$

$\Rightarrow \tan^{-1}\left[\frac{\sqrt{3}(x+y+1)}{2(1-x-y-2xy)}\right]=\frac{\sqrt{3}C}{2}$ **[1 Mark]**

$\Rightarrow \frac{\sqrt{3}(x+y+1)}{2(1-x-y-2xy)}=\tan\left(\frac{\sqrt{3}C}{2}\right)=B$, where $B=\tan\left(\frac{\sqrt{3}C}{2}\right)$

$$\Rightarrow x+y+1=\frac{2B}{\sqrt{3}}(1-xy-2xy)$$

$\Rightarrow x+y+1=A(1-x-y-2xy)$, where $A=\frac{2B}{\sqrt{3}}$

Hence proved **[1 Mark]**

8. Find the equation of the curve passing through the point $\left(0,\frac{\pi}{4}\right)$ whose differential equation is, $\sin x\cos y dx+\cos x\sin y dy=0$ **[4 Marks]**

Solution:

The differential equation of the given curve is $\sin x\cos y dx+\cos x\sin y dy=0$

$$\Rightarrow \frac{\sin x\cos y dx+\cos x\sin y dy}{\cos x\cos y}=0$$

$\Rightarrow \tan x dx+\tan y dy=0$ **[1 Mark]**

Integrating both sides, we get:

$\log(\sec x) + \log(\sec y) = \log C$

$\log(\sec x \cdot \sec y) = \log C$

$\Rightarrow \sec x \cdot \sec y = C \ldots (1)$ **[1 Mark]**

The curve passes through point $\left(0, \frac{\pi}{4}\right)$

$\therefore 1 \times \sqrt{2} = C$

$\Rightarrow C = \sqrt{2}$ **[1 Mark]**

On substituting $C = \sqrt{2}$

$\sec x \cdot \sec y = \sqrt{2}$

$$\Rightarrow \sec x \cdot \frac{1}{\cos y} = \sqrt{2}$$

$$\Rightarrow \cos y = \frac{\sec x}{\sqrt{2}}$$

Hence, the required equation of the curve is $\cos y = \frac{\sec x}{\sqrt{2}}$ **[1 Mark]**

9. Find the particular solution of the different equation\

$(1 + e^{2x})dy + (1 + y^2)e^x dx = 0$, given that $y = 1$ when $x = 0$ **[4 Marks]**

Solution:

Given differential equation is $(1 + e^{2x})dy + (1 + y^2)e^x dx = 0$

$$\Rightarrow \frac{dy}{1 + y^2} + \frac{e^x dx}{1 + e^{2x}} = 0$$

Integrating both sides, we get:

$\tan^{-1} y + \int \frac{e^x dx}{1+e^{2x}} = C \ldots (1)$ **[1 Mark]**

Let $e^x = t \Rightarrow e^{2x} = t^2$.

$$\Rightarrow \frac{d}{dx}(e^x) = \frac{dt}{dx}$$

$$\Rightarrow e^x = \frac{dt}{dx}$$

$\Rightarrow e^x dx = dt$ **[1 Mark]**

Substituting these values in equation (1), we get:

$$\tan^{-1}y + \int \frac{dt}{1+t^2} = C$$

$\Rightarrow \tan^{-1}y + \tan^{-1}t = C$

$\Rightarrow \tan^{-1}y + \tan^{-1}(e^x) = C \ldots (2)$ **[1 Mark]**

Now, $y = 1$ at $x = 0$.

Therefore, equation (2) becomes:

$\tan^{-1}1 + \tan^{-1}1 = C$

$\Rightarrow \frac{\pi}{4} + \frac{\pi}{4} = C$

$\Rightarrow C = \frac{\pi}{2}$

Substituting $C = \frac{\pi}{2}$ in equation (2), we get:

$$\tan^{-1}y + \tan^{-1}(e^x) = \frac{\pi}{2}$$

Hence, the required particular solution of the given differential equation is

$\tan^{-1}y + \tan^{-1}(e^x) = \frac{\pi}{2}$ **[1 Mark]**

10. Solve the differential equation $ye^{\frac{x}{y}}dx = \left(xe^{\frac{x}{y}} + y^2\right)dy (y \neq 0)$ **[2 Marks]**

Solution:

Given differential equation is $ye^{\frac{x}{y}}dx = \left(xe^{\frac{x}{y}} + y^2\right)dy$

$$\Rightarrow ye^{\frac{x}{y}}\frac{dx}{dy} = xe^{\frac{x}{y}} + y^2$$

$$\Rightarrow e^{\frac{x}{y}}\left[y \cdot \frac{dx}{dy} - x\right] = y^2$$

$\Rightarrow e^{\frac{x}{y}} \cdot \frac{\left[y \cdot \frac{dx}{dy} - x\right]}{y^2} = 1 \ldots (1)$ **[$\frac{1}{2}$ Mark]**

Let $e^{\frac{x}{y}} = z$

Differentiating it with respect to y, we get:

$$\frac{d}{dy}\left(e^{\frac{x}{y}}\right) = \frac{dz}{dy}$$

$$\Rightarrow e^{\frac{x}{y}} \cdot \frac{d}{dy}\left(\frac{x}{y}\right) = \frac{dz}{dy}$$

$\Rightarrow e^{\frac{x}{y}} \cdot \left[\frac{y\frac{dx}{dy}-x}{y^2}\right] = \frac{dz}{dy} \dots (2)$ **[$\frac{1}{2}$ Mark]**

From equation (1) and equation (2), we get:

$$\frac{dz}{dy} = 1$$

$$\Rightarrow dz = dy$$

Integrating both sides, we get:

$$z = y + C$$

$$\Rightarrow e^{\frac{x}{y}} = y + C$$

Hence, the required particular solution of the given differential equation is **[1 Mark]**

11. Find particular solution of the differential equation.

$(x - y)(dx + dy) = dx - dy$

Given that $y = -1$, when $x = 0$ (Hint: put $x - y = t$) **[4 Marks]**

Solution:

Given differential equation is $(x - y)(dx + dy) = dx - dy$

$$\Rightarrow (x - y + 1)dy = (1 - x + y)dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{1 - x + y}{x - y + 1}$$

$\Rightarrow \frac{dy}{dx} = \frac{1-(x-y)}{1+(x-y)} \dots (1)$ **[$\frac{1}{2}$ Mark]**

Let $x - y = t$

$\Rightarrow \frac{d}{dx}(x-y)=\frac{dt}{dx}$

$\Rightarrow 1-\frac{dy}{dx}=\frac{dt}{dx}$

$\Rightarrow 1-\frac{dt}{dx}=\frac{dy}{dx}$ **[$\frac{1}{2}$ Mark]**

Substituting the values of $x-y$ and $\frac{dy}{dx}$ in equation (1), we get:

$1-\frac{dt}{dx}=\frac{1-t}{1+t}$

$\Rightarrow \frac{dt}{dx}=1-\left(\frac{1-t}{1+t}\right)$

$\Rightarrow \frac{dt}{dx}=\frac{(1+t)-(1-t)}{1+t}$

$\Rightarrow \frac{dt}{dx}=\frac{2t}{1+t}$

$\Rightarrow \left(\frac{1+t}{t}\right)dt=2dx$

$\Rightarrow \left(1+\frac{1}{t}\right)dt=2dx \ldots(2)$ **[1 Mark]**

Integrating both sides, we get:

$t+\log|t|=2x+\text{C}$

$\Rightarrow (x-y)+\log|x-y|=2x+\text{C}$

$\Rightarrow \log|x-y|=x+y+\text{C} \ldots(3)$

Now, $y=-1$ at $x=0$.

Therefore, equation (3) becomes: $\log 1=0-1+C$

$\Rightarrow C=1$ **[1 Mark]**

Substituting $C=1$ in equation (3) we get:

$\log|x-y|=x+y+1$

Hence, the required particular solution of the given differential equation is

$\log|x-y|=x+y+1$ **[1 Mark]**

12. Solve the differential equation $\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right]\frac{dx}{dy}=1(x\neq 0)$ **[2 Marks]**

Solution:

Given differential equation is $\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right]\frac{dx}{dy}=1$

$\Rightarrow \frac{dy}{dx}=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}$

$\Rightarrow \frac{dy}{dx}+\frac{y}{\sqrt{x}}=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

The equation is a liner differential equation of the form

$\frac{dy}{dx}+Py=Q$,where $P=\frac{1}{\sqrt{x}}$ and $Q=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

Now, $\mathrm{I.F}=e^{\int(Q\times \mathrm{I.F.})}=e^{\int\frac{1}{\sqrt{x}}dx}=e^{2\sqrt{x}}$ **[1 Mark]**

The general solution of the given differential equation is given by,

$y(\mathrm{I.F.})=\int(\mathrm{Q}\times\mathrm{I.F.})\,dx+\mathrm{C}$

$\Rightarrow ye^{2\sqrt{x}}=\int\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}}\times e^{2\sqrt{x}}\right)dx+\mathrm{C}$

$\Rightarrow ye^{2\sqrt{x}}=\int\frac{1}{\sqrt{x}}dx+\mathrm{C}$

$\Rightarrow ye^{2\sqrt{x}}=2\sqrt{x}+\mathrm{C}$ **[1 Mark]**

13. Find a particular solution of the differential equation

$\frac{dy}{dx}+y\cot x=4x\operatorname{cosec}x(x\neq 0)$

Given that $y=0$ when $x=\frac{\pi}{2}$ **[2 Marks]**

Solution:

The given differential equation is $\frac{dy}{dx}+y\cot x=4x\operatorname{cosec}x$

This equation is a linear differential equation of the form

$\frac{dy}{dx}+py=Q$, where $p=\cot x$ and $Q=4x$ cosec.

Now, $I.F = e^{\int pdx} = e^{\int \cot x dx} = e^{\log|\sin x|} = \sin x$

[$\frac{1}{2}$ Mark]

The general solution of the given differential equation is given by,

$y(\text{I.F.}) = \int (\text{Q} \times \text{I.F.})\, dx + \text{C}$

$\Rightarrow y\sin x = \int (4x\text{cosec}x \cdot \sin x)\, dx + \text{C}$ [$\frac{1}{2}$ Mark]

$\Rightarrow y\sin x = 4\int x\, dx + \text{C}$

$\Rightarrow y\sin x = 4 \cdot \frac{x^2}{2} + \text{C}$

$\Rightarrow y\sin x = 2x^2 + \text{C} \dots (1)$

Now, $y = 0$ at $x = \frac{\pi}{2}$.

Therefore, equation (1) becomes:

$0 = 2 \times \frac{\pi^2}{4} + \text{C}$

$\Rightarrow C = -\frac{\pi^2}{2}$ [$\frac{1}{2}$ Mark]

Substituting $C = -\frac{\pi^2}{2}$

$y\sin x = 2x^2 - \frac{\pi^2}{2}$

Hence, the required particular solution of the given differential equation is

$y\sin x = 2x^2 - \frac{\pi^2}{2}$ [$\frac{1}{2}$ Mark]

14. Find a particular solution of the differential equation

$(x+1)\frac{dy}{dx} = 2e^{-y} - 1$

Given that $y = 0$ when $x = 0$ **[4 Marks]**

Solution:

Given differential equation is $(x+1)\frac{dy}{dx} = 2e^{-y} - 1$

$\Rightarrow \frac{dy}{2e^{-y} - 1} = \frac{dx}{x+1}$

$$\Rightarrow \frac{e^y dy}{2-e^y} = \frac{dx}{x+1}$$

Integrating both sides, we get:

$\int \frac{e^y dy}{2-e^y} = \log|x+1| + \log C \dots (1)$ **[1 Mark]**

Let $2 - e^y = t$

$$\therefore \frac{d}{dy}(2-e^y) = \frac{dt}{dy}$$

$$\Rightarrow -e^y = \frac{dt}{dy}$$

$$\Rightarrow -e^y dt = dt$$

Substituting this value in equation (1), we get:

$\int \frac{-dt}{t} = \log|x+1| + \log C$ **[1 Mark]**

$$\Rightarrow -\log|t| = \log|C(x+1)|$$

$$\Rightarrow -\log|2-e^y| = \log|C(x+1)|$$

$$\Rightarrow \frac{1}{2-e^y} = C(x+1)$$

$$\Rightarrow 2-e^y = \frac{1}{C(x+1)} \dots (2)$$

Now, at $x = 0$ and $y = 0$, equation (2) becomes:

$$\Rightarrow 2-1 = \frac{1}{C}$$

$\Rightarrow C = 1$ **[1 Mark]**

Substituting $C = 1$ in equation (2), we get:

$$2-e^y = \frac{1}{x+1}$$

$$\Rightarrow e^y = 2 - \frac{1}{x+1}$$

$$\Rightarrow e^y = \frac{2x+2-1}{x+1}$$

$$\Rightarrow e^y = \frac{2x+1}{x+1}$$

$$\Rightarrow y = \log\left|\frac{2x+1}{x+1}\right|, (x \neq -1)$$

Hence, the required particular solution of the given differential equation is

$y = \log\left|\frac{2x+1}{x+1}\right|, (x \neq -1)$ **[1 Mark]**

15. The population of a village increases continuously at the rate proportional to the number of its inhabitants present at any time. If the population of the village was 20000 in 1999 and 25000 in the year 2004, what will be the population of the village in 2009?

[4 Marks]

Solution:

Let the population at any instant (t) be y.

It is given that the rate of increase of population is proportional to the number of inhabitants at any instant.

$$\therefore \frac{dy}{dt} 0$$

$$\Rightarrow \frac{dy}{dt} = ky \text{ (} k \text{ is a constant)}$$

$$\Rightarrow \frac{dy}{y} = kdt$$

Integrating both sides, we get:

$$\log y = kt + C \ldots (1)$$

In the year 1999, $t = 0$ and $y = 20000$.

Therefore, we get:

$$\log 20000 = C \ldots (2)$$

In the year 2004, $t = 5$ and $y = 25000$.

Therefore, we get:

$$\log 25000 = k \cdot 5 + C$$ **[2 Marks]**

$$\Rightarrow \log 25000 = 5k + \log 20000$$

$$\Rightarrow 5k = \log\left(\frac{25000}{20000}\right) = \log\left(\frac{5}{4}\right)$$

$$\Rightarrow k = \frac{1}{5}\log\left(\frac{5}{4}\right) \ldots (3)$$ **[1 Mark]**

In the year 2009, $t = 10$ years.

Now, on substituting the values of t, k, and C in equation (1), we get:

$$\log y = 10 \times \frac{1}{5}\log\left(\frac{5}{4}\right) + \log(20000)$$

$$\Rightarrow \log y = \log\left[20000 \times \left(\frac{5}{4}\right)^2\right]$$

$\Rightarrow y = 20000 \times \frac{5}{4} \times \frac{5}{4}$

$\Rightarrow y = 31250$

Hence, the population of the village in 2009 will be 31250. **[1 Mark]**

16. The general solution of the differential equation $\frac{ydx-xdy}{y} = 0$ is **[2 Marks]**

A. $xy = C$

B. $x = Cy^2$

C. $y = Cx$

D. $y = Cx^2$

Solution:

The given differential equation is $\frac{ydx-xdy}{y} = 0$

$\Rightarrow \frac{ydx - xdy}{xy} = 0$

$\Rightarrow \frac{1}{x}dx - \frac{1}{y}dy = 0$ **[1 Mark]**

Integrating both sides, we get:

$\log|x| - \log|y| = \log k$

$\Rightarrow \log\left|\frac{x}{y}\right| = \log k$

$\Rightarrow \frac{x}{y} = k$

$\Rightarrow y = \frac{1}{k}x$

$\Rightarrow y = \mathrm{C}x$ where $\mathrm{C} = \frac{1}{k}$

Hence, the correct answer is C. **[1 Mark]**

17. The general solution of a differential equation of the type $\frac{dx}{dy} + \mathrm{P}_1 x = \mathrm{Q}_1$ is **[1 Mark]**

(A) $ye^{\int P_1 dy} = \int(Q_1 e^{\int Pdy})\, dy + C$

(B) $y \cdot e^{\int P_1 dx} = \int(Q_1 e^{\int P_1 dx})\, dy + C$

(C) $xe^{\int P_1 dy} = \int(Q_1 e^{\int P_1 dy})\, dy + C$

(D) $xe^{\int P_1 dx} = \int(Q_1 e^{\int P_1 dx})\, dy + C$

Solution:

The integrating factor of the given differential equation $\frac{dx}{dy} + P_1 x = Q_1$ is $e^{\int P_1 dy}$

The general solution of the differential equation is given by,

$x(\text{I.F.}) = \int(Q \times \text{I.F.})\, dy + C$ **[$\frac{1}{2}$ Mark]**

$\Rightarrow x \cdot e \int (Qe^{\int P_1 dy}) dy + C$

Hence, the correct answer is C. **[$\frac{1}{2}$ Mark]**

18. The general solution of the differential equation $e^x dy + (ye^x + 2x)dx = 0$ is **[2 Marks]**

(A) $xe^y + x^2 = C$

(B) $xe^y + y^2 = C$

(C) $ye^x + x^2 = C$

(D) $ye^y + x2 = C$

Solution:

The given differential equation is:

$e^x dy + (ye^x + 2x)dx = 0$

$\Rightarrow e^x \frac{dy}{dx} + ye^x + 2x = 0$

$\Rightarrow \frac{dy}{dx} + y = -2xe^{-x}$ **[$\frac{1}{2}$ Mark]**

This is a linear differential equation of the form

$\frac{dy}{dx} + Py = Q$, where $P = 1$ and $Q = -2xe^{-x}$

Now, $I.F = e^{\int Pdx} = e^{\int dx} = e^x$ **[$\frac{1}{2}$ Mark]**

The general solution of the given differential equation is given by,

$$y(\text{I.F.}) = \int (\text{Q} \times \text{I.F.})\, dx + C$$

$$\Rightarrow ye^{x} = \int (-2xe^{-x} \cdot e^{x})\, dx + C$$

$$\Rightarrow ye^{x} = -\int 2x\, dx + C$$

$$\Rightarrow ye^{x} = -x^{2} + C$$

$$\Rightarrow ye^{x} + x^{2} = C$$

Hence, the correct answer is C. **[1 Mark]**