CBSE NCERT Solutions for Class 12 Maths Chapter 07

Back of Chapter Questions

Exercise 7.1

1. Find an anti-derivative (or integral) of the function $\sin 2x$ by the method of inspection.

[2 Marks]

Solution:

Anti-derivative of $\sin 2x$ is a function of x whose derivative is $\sin 2x$.

As we know that, $\frac{d}{dx}(\cos 2x) = -2\sin 2x$ **[1 Mark]**

$\sin 2x = -\frac{1}{2}\frac{d}{dx}(\cos 2x)$

$\therefore \sin 2x = \frac{d}{dx}\left(-\frac{1}{2}\cos 2x\right)$

Hence, the anti − derivative of $\sin 2x$ is $-\frac{1}{2}\cos 2x$ **[1 Mark]**

2. Find an anti-derivative (or integral) of the function $\cos 3x$ by the method of inspection.

[2 Marks]

Solution:

Anti-derivative of $\cos 3x$ is a function of x whose derivative is $\cos 3x$.

As we know that, $\frac{d}{dx}(\sin 3x) = 3\cos 3x$ **[1 Mark]**

$\Rightarrow \cos 3x = \frac{1}{3}\frac{d}{dx}(\sin 3x)$

$\therefore \cos 3x = \frac{d}{dx}\left(\frac{1}{3}\sin 3x\right)$

Hence, the anti-derivative of $\cos 3x$ is $\frac{1}{3}\sin 3x$ **[1 Mark]**

3. Find an anti-derivative (or integral) of the function e^{2x} by the method of inspection.

[2 Marks]

Solution:

Anti-derivative of e^{2x} is the function of x whose derivative is e^{2x}.

As we know that, $\frac{d}{dx}(e^{2x}) = 2e^{2x}$ **[1 Mark]**

$$\Rightarrow e^{2x} = \frac{1}{2}\frac{d}{dx}(e^{2x})$$

$\therefore e^{2x} = \frac{1}{2}\frac{d}{dx}(e^{2x})$

Hence, the anti-derivative of e^{2x} is $\frac{1}{2}e^{2x}$ **[1 Mark]**

4. Find an anti-derivative (or integral) of the function $(ax+b)^2$ by the method of inspection.

[2 Marks]

Solution:

Anti-derivative of $(ax+b)^2$ is the function of x whose derivative is $(ax+b)^2$.

As we know that, $\frac{d}{dx}(ax+b)^3 = 3a(ax+b)^2$ **[1 Mark]**

$$\Rightarrow (ax+b)^2 = \frac{1}{3a}\frac{d}{dx}(ax+b)^3$$

$\therefore (ax+b)^2 = \frac{d}{dx}\left(\frac{1}{3a}(ax+b)^3\right)$

Hence, the anti-derivative of $(ax+b)^2$ is $\frac{1}{3a}(ax+b)^3$ **[1 Mark]**

5. Find an anti-derivative (or integral) of the function $\sin 2x - 4e^{3x}$ by the method of inspection.

[2 Marks]

Solution:

Anti-derivative of $\sin 2x - 4e^{3x}$ is the function of x whose derivative is $\sin 2x - 4e^{3x}$

As we know that, $\frac{d}{dx}\left(-\frac{1}{2}\cos 2x - \frac{4}{3}e^{3x}\right) = \sin 2x - 4e^{3x}$ **[1 Mark]**

Hence, the anti-derivative of $(\sin 2x - 4e^{2x})$ is $\left(-\frac{1}{2}\cos 2x - \frac{4}{3}e^{3x}\right)$ **[1 Mark]**

6. Find the integral $\int(4e^{3x} + 1)dx$. **[2 Marks]**

Solution:

The given integral is $\int(4e^{3x} + 1)dx$

$= 4\int e^{3x}dx + \int 1dx$

$= 4\left(\frac{e^{3x}}{3}\right) + x + C \quad [\because \int e^{ax}dx = \frac{e^{ax}}{a} + C \text{ and } \int a\,dx = ax + C]$ **[1 Mark]**

$= \frac{4}{3}e^{3x} + x + C$

Hence, integral $\int(4e^{3x} + 1)dx = \frac{4}{3}e^{3x} + x + C$. **[1 Mark]**

7. Find the integral $\int x^2\left(1 - \frac{1}{x^2}\right)dx$ **[2 Marks]**

Solution:

The given integral is $\int x^2\left(1 - \frac{1}{x^2}\right)dx$

$= \int(x^2 - 1)dx$

$= \int x^2dx - \int 1dx$

[$\frac{1}{2}$ Mark]

$= \frac{x^3}{3} - x + C \quad [\because \int x^n dx = \frac{x^{n+1}}{n+1} + C \text{ and } \int a\,dx = ax + C]$ **[1 Mark]**

Hence, the integral $\int x^2\left(1 - \frac{1}{x^2}\right)dx = \frac{x^3}{3} - x + C$

[$\frac{1}{2}$ Mark]

8. Find the integral $\int(ax^2+bx+c)dx$ **[2 Marks]**

Solution:

The given integral is $\int(ax^2+bx+c)dx$

$= a\int x^2dx + b\int xdx + c\int 1.dx$

$= a\left(\frac{x^3}{3}\right) + b\left(\frac{x^2}{2}\right) + cx + C \qquad [\because \int x^n dx = \frac{x^{n+1}}{n+1} + C \text{ and } \int a\,dx = ax + C]$

[1 Mark]

$= \frac{ax^3}{3} + \frac{bx^2}{2} + cx + C$

Hence, the integral $\int(ax^2+bx+c)dx = \frac{ax^3}{3} + \frac{bx^2}{2} + cx + C$

[1 Mark]

9. Find the integral $\int(2x^2+e^x)\,dx$. **[2 Marks]**

Solution:

The given integral is $\int(2x^2+e^x)\,dx$

$= 2\int x^2dx + \int e^x dx$

$= 2\left(\frac{x^3}{3}\right) + e^x + C \qquad [\because \int x^n dx = \frac{x^{n+1}}{n+1} + C \text{ and } \int e^{ax}dx = \frac{e^{ax}}{a} + C]$

[1 Mark]

$= \frac{2}{3}x^3 + e^x + C$

Hence, the integral $\int(2x^2+e^x)\,dx = \frac{2}{3}x^3 + e^x + C.$

[1 Mark]

10. Find the integral $\int\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$ **[2 Marks]**

Solution:

The given integral is $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$

$= \int \left(x + \frac{1}{x} - 2\right) dx$

$= \int x dx + \int \frac{1}{x} dx - 2 \int 1. dx$

[$\frac{1}{2}$ Mark]

$= \frac{x^2}{2} + \log|x| - 2x + C$ $[\because \int x^n dx = \frac{x^{n+1}}{n+1} + C, \int a\, dx = ax + C \text{ and } \int \frac{1}{x} dx = \log|x| + C]$

[1 Mark]

Hence, the integral $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx = \frac{x^2}{2} + \log|x| - 2x + C$

[$\frac{1}{2}$ Mark]

11. Find the integral $\int \frac{x^3+5x^2-4}{x^2} dx$ **[2 Marks]**

Solution:

The given integral is $\int \frac{x^3+5x^2-4}{x^2} dx$

$= \int (x + 5 - 4x^{-2}) dx$

$= \int x\, dx + 5 \int 1. dx - 4 \int x^{-2} dx$

[$\frac{1}{2}$ Mark]

$= \frac{x^2}{2} + 5x - 4\left(\frac{x^{-1}}{-1}\right) + C$ $[\because \int x^n dx = \frac{x^{n+1}}{n+1} + C]$

[1 Mark]

$= \frac{x^2}{2} + 5x + \frac{4}{x} + C$

Hence, the integral $\int \frac{x^3+5x^2-4}{x^2} dx = \frac{x^2}{2} + 5x + \frac{4}{x} + C$

[$\frac{1}{2}$ Mark]

12. Find the integral $\int \frac{x^3+3x+4}{\sqrt{x}} dx$ **[2 Marks]**

Solution:

The given integral is $\int \frac{x^3+3x+4}{\sqrt{x}} dx$

$= \int \left(x^{\frac{5}{2}} + 3x^{\frac{1}{2}} + 4x^{-\frac{1}{2}}\right) dx$

$= \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{3\left(x^{\frac{3}{2}}\right)}{\frac{3}{2}} + \frac{4\left(x^{\frac{1}{2}}\right)}{\frac{1}{2}} + C$ $[\because \int x^n dx = \frac{x^{n+1}}{n+1} + C]$ **[1 Mark]**

$= \frac{2}{7}x^{\frac{7}{2}} + 2x^{\frac{3}{2}} + 8x^{\frac{1}{2}} + C$

$= \frac{2}{7}x^{\frac{7}{2}} + 2x^{\frac{3}{2}} + 8\sqrt{x} + C$

Hence, the integral $\int \frac{x^3+3x+4}{\sqrt{x}} dx = \frac{2}{7}x^{\frac{7}{2}} + 2x^{\frac{3}{2}} + 8\sqrt{x} + C$ **[1 Mark]**

13. Find the integral $\int \frac{x^3-x^2+x-1}{x-1} dx$ **[2 Marks]**

Solution:

The given integral is $\int \frac{x^3-x^2+x-1}{x-1} dx$

On dividing, we get

$= \int (x^2 + 1) dx$ **[$\frac{1}{2}$ Mark]**

$= \int x^2 dx + \int 1 dx$

$= \frac{x^3}{3} + x + C$ $[\because \int x^n dx = \frac{x^{n+1}}{n+1} + C \text{ and } \int a\, dx = ax + C]$ **[1 Mark]**

Hence, the integral $\int \frac{x^3-x^2+x-1}{x-1} dx = \frac{x^3}{3} + x + C$ **[$\frac{1}{2}$ Mark]**

14. Find the integral $\int (1-x)\sqrt{x} dx$ **[2 Marks]**

Solution:

The given integral is $\int(1-x)\sqrt{x}dx$

$= \int\left(\sqrt{x} - x^{\frac{3}{2}}\right)dx$

[$\frac{1}{2}$ Mark]

$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + C$ [$\because \int x^n dx = \frac{x^{n+1}}{n+1} + C$] **[1 Mark]**

$= \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} + C$ **[1 Mark]**

Hence, the integral $\int(1-x)\sqrt{x}dx = \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} + C$

[$\frac{1}{2}$ Mark]

15. Find the integral $\int \sqrt{x}(3x^2 + 2x + 3)dx$ **[2 Marks]**

Solution:

The given integral is $\int \sqrt{x}(3x^2 + 2x + 3)dx$

$= \int\left(3x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + 3x^{\frac{1}{2}}\right)dx$

[$\frac{1}{2}$ Mark]

$= 3\left(\frac{x^{\frac{7}{2}}}{\frac{7}{2}}\right) + 2\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right) + \frac{3\left(x^{\frac{3}{2}}\right)}{\frac{3}{2}} + C$ [$\because \int x^n dx = \frac{x^{n+1}}{n+1} + C$] **[1 Mark]**

$= \frac{6}{7}x^{\frac{7}{2}} + \frac{4}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + C$

Hence, the integral $\int \sqrt{x}(3x^2 + 2x + 3)dx = \frac{6}{7}x^{\frac{7}{2}} + \frac{4}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + C$

[$\frac{1}{2}$ Mark]

16. Find the integral $\int(2x - 3\cos x + e^x)dx$ **[4 Marks]**

Solution:

The given integral is $\int(2x - 3\cos x + e^x)dx$

$= 2\int x dx - 3\int \cos x\, dx + \int e^x dx$

[$\frac{1}{2}$ Mark]

$= \frac{2x^2}{2} - 3(\sin x) + e^x + C$ **[3 Marks]**

$[\because \int x^{nx}dx = \frac{x^{n+1}}{n+1} + C, \int e^{ax}dx = \frac{e^{ax}}{a} + C$ and $\int cosx\, dx = sinx + C]$

$= x^2 - 3\sin x + e^x + C$

Hence, the integral $\int(2x - 3\cos x + e^x)dx = x^2 - 3\sin x + e^x + C$

[$\frac{1}{2}$ Mark]

17. Find the integral $\int\left(2x^2 - 3\sin x + 5\sqrt{x}\right)dx$ **[4 Marks]**

Solution:

The given integral is $\int\left(2x^2 - 3\sin x + 5\sqrt{x}\right)dx$

$= 2\int x^2 dx - 3\int \sin x dx + 5\int x^{\frac{1}{2}}dx$

[$\frac{1}{2}$ Mark]

$= \frac{2x^3}{3} - 3(-\cos x) + 5\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$

$[\because \int x^n dx = \frac{x^{n+1}}{n+1} + C,$ and $\int \sin x\, dx = -\cos x + C]$ **[3 Marks]**

$= \frac{2}{3}x^3 + 3\cos x + \frac{10}{3}x^{\frac{3}{2}} + C$

Hence, the integral $\int\left(2x^2 - 3\sin x + 5\sqrt{x}\right)dx = \frac{2}{3}x^3 + 3\cos x + \frac{10}{3}x^{\frac{3}{2}} + C$

[$\frac{1}{2}$ Mark]

18. Find the integral $\int \sec x(\sec x + \tan x)dx$ **[2 Marks]**

Solution:

The given integral is $\int \sec x(\sec x + \tan x)dx$

$= \int(\sec^2 x + \sec x \tan x)dx$

[$\frac{1}{2}$ Mark]

$= \int \sec^2 x dx + \int \sec x \tan x dx$

$[\because \int \sec x \tan x \ dx = \sec x + C \ \ and \int \sec^2 x \ dx = \tan x + C]$

[1 Mark]

$= \tan x + \sec x + C$

Hence, the integral $\int \sec x(\sec x + \tan x)dx = \tan x + \sec x + C$

[$\frac{1}{2}$ Mark]

19. Find the integral $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx$ **[3 Marks]**

Solution:

The given integral is $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx$

$= \int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}} dx$

$= \int \frac{\sin^2 x}{\cos^2 x} dx$

$= \int \tan^2 x \, dx \qquad [\because \int \frac{\sin x}{\cos x} = \tan x]$

[1 Mark]

$= \int(\sec^2 x - 1)dx \qquad [\because \tan^2 x = \sec^2 x - 1]$

[$\frac{1}{2}$ Mark]

$= \int \sec^2 x dx - \int 1 dx$

$= \tan x - x + C \qquad [\because \int \sec^2 x dx = \tan x + C]$

[1 Marks]

Hence, the integral $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx = \tan x - x + C$

[$\frac{1}{2}$ Mark]

20. Find the integral $\int \frac{2-3\sin x}{\cos^2 x} dx$ **[4 Marks]**

Solution:

The given integral is $\int \frac{2-3\sin x}{\cos^2 x} dx$

$= \int \left(\frac{2}{\cos^2 x} - \frac{3\sin x}{\cos^2 x}\right) dx$

$\int 2\sec^2 x dx - 3\int \tan x \sec x dx$ $\quad$ $[\because \frac{\sin x}{\cos x} = \tan x \; and \; \frac{1}{\cos x} = \sec x]$

[1 Mark]

$= 2\tan x - 3\sec x + C$

[2 Marks]

$[\because \int \sec^2 x \; dx = \tan \; x + C \;\; and \; \int \tan x \sec x dx = \; \sec x + C]$

Hence, the integral $\int \frac{2-3\sin x}{\cos^2 x} dx = 2\tan x - 3\sec x + C$

[1 Mark]

21. The anti-derivative of $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$ equals

(A) $\frac{1}{3}x^{\frac{1}{3}} + 2x^{\frac{1}{2}} + C$

(B) $\frac{2}{3}x^{\frac{2}{3}} + \frac{1}{2}x^2 + C$

(C) $\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$

(D) $\frac{3}{2}x^{\frac{3}{2}} + \frac{1}{2}x^{\frac{1}{2}} + C$ **[2 Marks]**

Solution:

$\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$

$= \int x^{\frac{1}{2}} dx + \int x^{-\frac{1}{2}} dx$

$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$ $\quad [\because \int x^n dx = \frac{x^{n+1}}{n+1} + C]$

$= \frac{2}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$

The anti-derivative of $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) = \frac{2}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$ **[2 Marks]**

Hence, the correct answer is (C)

22. If $\frac{d}{dx} f(x) = 4x^3 - \frac{3}{x^4}$ such that $f(2) = 0$. Then $f(x)$ is

(A) $x^4 + \frac{1}{x^3} - \frac{129}{8}$

(B) $x^3 + \frac{1}{x^4} + \frac{129}{8}$

(C) $x^4 + \frac{1}{x^3} + \frac{129}{8}$

(D) $x^3 + \frac{1}{x^4} - \frac{129}{8}$ **[4 Marks]**

Solution:

Given that, $\frac{d}{dx} f(x) = 4x^3 - \frac{3}{x^4}$

Thus, anti-derivative of $4x^3 - \frac{3}{x^4} = f(x)$ **[1 Mark]**

$\therefore f(x) = \int 4x^3 - \frac{3}{x^4} dx$

$\Rightarrow f(x) = 4 \int x^3 dx - 3 \int (x^{-4}) dx$

$\Rightarrow f(x) = 4\left(\frac{x^4}{4}\right) - 3\left(\frac{x^{-3}}{-3}\right) + C$ $\quad [\because \int x^n dx = \frac{x^{n+1}}{n+1} + C]$ **[1 Mark]**

$\Rightarrow f(x) = x^4 + \frac{1}{x^3} + C$

Also,

$f(2) = 0$

$\therefore f(2) = (2)^4 + \frac{1}{(2)^3} + C = 0$

$\Rightarrow 16 + \frac{1}{8} + C = 0$

$\Rightarrow C = -\left(16 + \frac{1}{8}\right)$

$\Rightarrow C = \frac{-129}{8}$

[1 Mark]

$\therefore f(x) = x^4 + \frac{1}{x^3} - \frac{129}{8}$ **[1 Mark]**

Hence, the correct answer is (A)

Exercise 7.2

1. Integrate the function $\frac{2x}{1+x^2}$ **[4 Marks]**

Solution:

We need to integrate $\frac{2x}{1+x^2}$

Let $1 + x^2 = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$2x\frac{dx}{dt} = 1$

$\therefore 2x\ dx = dt$ **[1 Mark]**

$\Rightarrow \int \frac{2x}{1+x^2} dx = \int \frac{1}{t} dt$

[$\frac{1}{2}$ Mark]

$= \log|t| + C$ $\quad [\because \int \frac{1}{x} dx = \log|x| + C\ ]$

[1 Mark]

$= \log|1 + x^2| + C$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function $\frac{2x}{1+x^2} = \log(1+x^2) + C$

[$\frac{1}{2}$ Mark]

2. Integrate the function $\frac{(\log x)^2}{x}$ **[4 Marks]**

Solution:

We need to integrate $\frac{(\log x)^2}{x}$

Let $\log|x| = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$\frac{1}{x}\frac{dx}{dt} = 1$

$\therefore \frac{1}{x}dx = dt$ **[1 Mark]**

$\Rightarrow \int \frac{(\log|x|)^2}{x}dx = \int \frac{(t)^2}{x}dx$ **[$\frac{1}{2}$ Mark]**

$= \int (t)^2 dt$

$= \frac{t^3}{3} + C$ $\quad [\because \int x^n dx = \frac{x^{n+1}}{n+1} + C]$ **[1 Mark]**

$= \frac{(\log|x|)^3}{3} + C$ $\quad$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function $\frac{(\log x)^2}{x} = \frac{(\log|x|)^3}{3} + C.$

[$\frac{1}{2}$ Mark]

3. Integrate the function $\frac{1}{x+x\log x}$ **[4 Marks]**

Solution:

We need to integrate $\frac{1}{x+x\log x}$

$= \frac{1}{x(1+\log x)}$

Let $1 + \log x = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$\frac{1}{x}\frac{dx}{dt} = 1$

$\therefore \frac{1}{x}dx = dt$ **[1 Mark]**

$\Rightarrow \int \frac{1}{x(1+\log x)}dx \ = \int \frac{1}{t}dt$ $[\because \int \frac{1}{x}dx = \log x + C]$

[$\frac{1}{2}$ Mark]

$= \log|t| + C$

[$\frac{1}{2}$ Mark]

$= \log|1 + \log x| + C$ [substituting t] **[1 Mark]**

Hence, integration of the function $\frac{1}{x+x\log x} = = \log|1 + \log x| + C$

[$\frac{1}{2}$ Mark]

4. Integrate the function $\sin x \sin(\cos x)$ **[4 Marks]**

Solution:

We need to integrate $\sin x \sin(\cos x)$

Let $\cos x = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$-\sin x \frac{dx}{dt} = 1$

$\therefore -\sin x\ dx = dt$ **[1 Mark]**

$\Rightarrow \int \sin x \sin(\cos x)dx = -\int \sin t\ dt$

[$\frac{1}{2}$ Mark]

$= -[-\cos t] + C$ $[\because \int \sin x\, \mathrm{dx} = -\cos x + C]$ **[1 Mark]**

$= \cos t + C$

$= \cos(\cos x) + C$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function $\sin x \sin(\cos x) = \cos(\cos x) + C$

[$\frac{1}{2}$ Mark]

5. Integrate the function $\sin(ax+b)\cos(ax+b)$ **[4 Marks]**

Solution:

We need to integrate $\sin(ax+b)\cos(ax+b)$

$= \frac{2\sin(ax+b)\cos(ax+b)}{2}$ [$\because$ multiplying and dividing it by 2]

$= \frac{\sin 2(ax+b)}{2}$ [$\because 2\sin(x)\cos(x) = \sin 2(x)$]

[$\frac{1}{2}$ Mark]

Let $2(ax+b) = t$

Differentiating with respect to t, we get

$2a\frac{dx}{dt} = 1$

$\therefore 2adx = dt$ **[1 Mark]**

$$\Rightarrow \int \frac{\sin 2(ax+b)}{2} dx = \frac{1}{2}\int \frac{\sin t \, dt}{2a}$$

$= \frac{1}{4a}[-\cos t] + C$ [$\because \int \sin x \, dx = -\cos x + C$] **[1 Mark]**

$= -\frac{1}{4a}\cos 2(ax+b) + C$ [substituting t] **[1 Mark]**

Hence, integration of the function $\sin(ax+b)\cos(ax+b) = -\frac{1}{4a}\cos 2(ax+b) + C$

[$\frac{1}{2}$ Mark]

6. Integrate the function $\sqrt{ax+b}$ **[4 Marks]**

Solution:

We need to integrate $\sqrt{ax+b}$

Let $ax+b=t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$a\frac{dx}{dt}=1$

$\Rightarrow adx=dt$

$\therefore dx=\frac{1}{a}dt$

[1 Mark]

$\Rightarrow \int(ax+b)^{\frac{1}{2}}dx=\frac{1}{a}\int t^{\frac{1}{2}}dt$

[$\frac{1}{2}$ Mark]

$=\frac{1}{a}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right)+C$ $\quad$ $[\because \int x^n dx= \frac{x^{n+1}}{n+1}+C]$

[1 Mark]

$=\frac{2}{3a}(ax+b)^{\frac{3}{2}}+C$ $\quad$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function $\sqrt{ax+b}$ $=\frac{2}{3a}(ax+b)^{\frac{3}{2}}+C$

[$\frac{1}{2}$ Mark]

7. Integrate the function $x\sqrt{x+2}$ **[4 Marks]**

Solution:

We need to integrate $x\sqrt{x+2}$

Let $x+2=t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$\frac{dx}{dt}=1$

$\therefore dx = dt$

[1 Mark]

$\Rightarrow \int x\sqrt{x+2}dx = \int (t-2)\sqrt{t}dt$

[$\frac{1}{2}$ Mark]

$= \int \left(t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\right) dt$

$= \int t^{\frac{3}{2}} dt - 2\int t^{\frac{1}{2}} dt$

$= \frac{t^{\frac{5}{2}}}{\frac{5}{2}} - 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C \qquad [\because \int x^n dx = \frac{x^{n+1}}{n+1} + C]$

[1 Mark]

$= \frac{2}{5}t^{\frac{5}{2}} - \frac{4}{3}t^{\frac{3}{2}} + C$

$= \frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}} + C$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function, $x\sqrt{x+2} = \frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}} + C$

[$\frac{1}{2}$ Mark]

8. Integrate the function $x\sqrt{1+2x^2}$
[4 Marks]

Solution:

We need to integrate $x\sqrt{1+2x^2}$

Let $1 + 2x^2 = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$4x\frac{dx}{dt} = 1$

$\therefore 4xdx = dt$

[1 Mark]

$\Rightarrow \int x\sqrt{1+2x^2}\,dx = \int \frac{\sqrt{t}dt}{4}$

[$\frac{1}{2}$ Mark]

$= \frac{1}{4}\int t^{\frac{1}{2}}dt$

$= \frac{1}{4}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$ $\quad [\because \int x^n dx = \frac{x^{n+1}}{n+1} + C]$

[1 Mark]

$= \frac{1}{6}(1+2x^2)^{\frac{3}{2}} + C$ $\quad$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function, $x\sqrt{1+2x^2} = \frac{1}{6}(1+2x^2)^{\frac{3}{2}} + C$

[$\frac{1}{2}$ Mark]

9. Integrate the function $(4x+2)\sqrt{x^2+x+1}$ **[4 Marks]**

Solution:

We need to integrate $(4x+2)\sqrt{x^2+x+1}$

Let $x^2+x+1 = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$(2x+1)\frac{dx}{dt} = 1$

$\therefore (2x+1)dx = dt$ **[1 Mark]**

$\int (4x+2)\sqrt{x^2+x+1}\,dx$

$= \int 2\sqrt{t}\,dt$

[$\frac{1}{2}$ Mark]

$= 2\int \sqrt{t}\,dt$

$= 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$ $\quad [\because \int x^n dx = \frac{x^{n+1}}{n+1} + C]$ **[1 Mark]**

$= \frac{4}{3}(x^2 + x + 1)^{\frac{3}{2}} + C$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function, $(4x + 2)\sqrt{x^2 + x + 1} = \frac{4}{3}(x^2 + x + 1)^{\frac{3}{2}} + C$

[$\frac{1}{2}$ Mark]

10. Integrate the function $\frac{1}{x-\sqrt{x}}$ **[4 Marks]**

Solution:

We need to integrate $\frac{1}{x-\sqrt{x}}$

$$= \frac{1}{\sqrt{x}\left(\sqrt{x} - 1\right)}$$

Let $\left(\sqrt{x} - 1\right) = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$\frac{1}{2\sqrt{x}}\frac{dx}{dt} = 1$

$\therefore \frac{1}{2\sqrt{x}}dx = dt$ **[1 Mark]**

$$\Rightarrow \int \frac{1}{\sqrt{x}\left(\sqrt{x} - 1\right)}dx$$

$= \int \frac{2}{t}dt$

[$\frac{1}{2}$ Mark]

$= 2\log|t| + C$ $\quad [\because \int \frac{1}{x}dx = \log x + C]$ **[1 Mark]**

$= 2\log\left|\sqrt{x} - 1\right| + C$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function, $\frac{1}{x-\sqrt{x}} = 2\log\left|\sqrt{x} - 1\right| + C$

[$\frac{1}{2}$ Mark]

11. Integrate the function $\frac{x}{\sqrt{x+4}}, x > 0$ **[4 Marks]**

Solution:

We need to integrate $\frac{x}{\sqrt{x+4}}$

Let $x + 4 = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$\frac{dx}{dt} = 1$

$\therefore dx = dt$ **[1 Mark]**

$$= \int \frac{x}{\sqrt{x+4}} dx$$

$= \int \frac{(t-4)}{\sqrt{t}} dt$

[$\frac{1}{2}$ Mark]

$$= \int \left(\sqrt{t} - \frac{4}{\sqrt{t}}\right) dt$$

$$= \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - 4\left(\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right) + C \qquad \left[\because \int x^n dx = \frac{x^{n+1}}{n+1} + C\right]$$ **[1 Mark]**

$$= \frac{2}{3}(t)^{\frac{3}{2}} - 8(t)^{\frac{1}{2}} + C$$

$$= \frac{2}{3} t \cdot t^{\frac{1}{2}} - 8t^{\frac{1}{2}} + C$$

$$= \frac{2}{3} t^{\frac{1}{2}}(t - 12) + C$$

$= \frac{2}{3}(x+4)^{\frac{1}{2}}(x + 4 - 12) + C$ [substituting t]

[$\frac{1}{2}$ Mark]

$$= \frac{2}{3}\sqrt{x+4}(x - 8) + C$$

Hence, integration of the function $\frac{x}{\sqrt{x+4}} = \frac{2}{3}\sqrt{x+4}(x-8) + C$

[$\frac{1}{2}$ Mark]

12. Integrate the function $(x^3-1)^{\frac{1}{3}}x^5$ **[4 Marks]**

Solution:

We need to integrate $(x^3-1)^{\frac{1}{3}}x^5$

Let $x^3-1=t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$3x^2\frac{dx}{dt}=1$

$\therefore 3x^2dx=dt$ **[1 Mark]**

$\Rightarrow \int (x^3-1)^{\frac{1}{3}}x^5dx=\int (x^3-1)^{\frac{1}{3}}x^3\cdot x^2dx$

$=\int t^{\frac{1}{3}}(t+1)\frac{dt}{3}$

[$\frac{1}{2}$ Mark]

$=\frac{1}{3}\left[\frac{t^{\frac{7}{3}}}{\frac{7}{3}}+\frac{t^{\frac{4}{3}}}{\frac{4}{3}}\right]+C$ $[\because \int x^n dx=\frac{x^{n+1}}{n+1}+C]$ **[1 Mark]**

$=\frac{1}{3}\left[\frac{3}{7}t^{\frac{7}{3}}+\frac{3}{4}t^{\frac{4}{3}}\right]+C$

$=\frac{1}{7}(x^3-1)^{\frac{7}{3}}+\frac{1}{4}(x^3-1)^{\frac{4}{3}}+C$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function $(x^3-1)^{\frac{1}{3}}x^5=\frac{1}{7}(x^3-1)^{\frac{7}{3}}+\frac{1}{4}(x^3-1)^{\frac{4}{3}}+C$

[$\frac{1}{2}$ Mark]

13. Integrate the function $\frac{x^2}{(2+3x^3)^3}$ **[4 Marks]**

Solution:

We need to integrate $\frac{x^2}{(2+3x^3)^3}$

Let $2 + 3x^3 = t$

[$\frac{1}{2}$ **Mark]**

Differentiating with respect to t, we get

$9x^2 \frac{dx}{dt} = 1$

$\therefore 9x^2 dx = dt$ **[1 Mark]**

$\Rightarrow \int \frac{x^2}{(2+3x^3)^3} dx = \frac{1}{9} \int \frac{dt}{(t)^3}$ [$\frac{1}{2}$ **Mark]**

$= \frac{1}{9}\left[\frac{t^{-2}}{-2}\right] + C$ $\quad [\because \int x^n dx = \frac{x^{n+1}}{n+1} + C]$ **[1 Mark]**

$= \frac{-1}{18}\left(\frac{1}{t^2}\right) + C$

$= \frac{-1}{18(2+3x^3)^2} + C$ [substituting t]

[$\frac{1}{2}$ **Mark]**

Hence, integration of the function $\frac{x^2}{(2+3x^3)^3} = \frac{-1}{18(2+3x^3)^2} + C$

[$\frac{1}{2}$ **Mark]**

14. Integrate the function $\frac{1}{x(\log x)^m}, x > 0, m \neq 1$ **[4 Marks]**

Solution:

We need to integrate $\frac{1}{x(\log x)^m}$

Let $\log x = t$

[$\frac{1}{2}$ **Mark]**

Differentiating with respect to t, we get

$\frac{1}{x}\frac{dx}{dt} = 1$

$\frac{1}{x} dx = dt$ **[1 Mark]**

$\Rightarrow \int \frac{1}{x(\log x)^m} dx$

$= \int \frac{dt}{(t)^m}$

[$\frac{1}{2}$ Mark]

$= \left(\frac{t^{-m+1}}{1-m}\right) + C$ [$\because \int x^n dx = \frac{x^{n+1}}{n+1} + C$] **[1 Mark]**

$= \frac{(\log x)^{1-m}}{(1-m)} + C$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function $\frac{1}{x(\log x)^m} = \frac{(\log x)^{1-m}}{(1-m)} + C$

[$\frac{1}{2}$ Mark]

15. Integrate the function $\frac{x}{9-4x^2}$ **[4 Marks]**

Solution:

We need to integrate $\frac{x}{9-4x^2}$

Let $9 - 4x^2 = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$-8x\frac{dx}{dt} = 1$

$\therefore -8x\, dx = dt$ **[1 Mark]**

$\Rightarrow \int \frac{x}{9-4x^2} dx$

$= \frac{-1}{8}\int \frac{1}{t} dt$

[$\frac{1}{2}$ Mark]

$= \frac{-1}{8}\log|t| + C$ [$\because \int \frac{1}{x} dx = \log x + C$] **[1 Mark]**

$= \frac{-1}{8}\log|9-4x^2| + C$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function $\frac{x}{9-4x^2} = \frac{-1}{8}\log|9-4x^2| + C$

[$\frac{1}{2}$ **Mark]**

16. Integrate the function e^{2x+3} **[4 Marks]**

Solution:

We need to integrate e^{2x+3}

Let $2x + 3 = t$

[$\frac{1}{2}$ **Mark]**

Differentiating with respect to t, we get

$2\frac{dx}{dt} = 1$

$\therefore 2dx = dt$ **[1 Mark]**

$\Rightarrow \int e^{2x+3}dx$

$= \frac{1}{2}\int e^t dt$

[$\frac{1}{2}$ **Mark]**

$= \frac{1}{2}(e^t) + C \quad [\because \int e^x dx = e^x + C]$ **[1 Mark]**

$= \frac{1}{2}e^{(2x+3)} + C \quad$ [substituting t]

[$\frac{1}{2}$ **Mark]**

Hence, integration of the function $e^{2x+3} = \frac{1}{2}e^{(2x+3)} + C$

[$\frac{1}{2}$ **Mark]**

17. Integrate the function $\frac{x}{e^{x^2}}$ **[4 Marks]**

Solution:

We need to integrate $\frac{x}{e^{x^2}}$

Let $x^2 = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$2x\frac{dx}{dt} = 1$

$\therefore 2xdx = dt$ **[1 Mark]**

$\Rightarrow \int \frac{x}{e^{x^2}}dx = \frac{1}{2}\int \frac{1}{e^t}dt$

[$\frac{1}{2}$ Mark]

$= \frac{1}{2}\int e^{-t}\,dt$

$= \frac{1}{2}\left(\frac{e^{-t}}{-1}\right) + C$ $\quad [\because \int e^{ax}dx = \frac{e^{ax}}{a} + C]$ **[1 Mark]**

$= -\frac{1}{2}e^{-x^2} + C$ $\quad$ [substituting t]

[$\frac{1}{2}$ Mark]

$= \frac{-1}{2e^{x^2}} + C$

Hence, integration of the function $\frac{x}{e^{x^2}} = \frac{-1}{2e^{x^2}} + C$

[$\frac{1}{2}$ Mark]

18. Integrate the function $\frac{e^{\tan^{-1}x}}{1+x^2}$ **[4 Marks]**

Solution:

We need to integrate $\frac{e^{\tan^{-1}x}}{1+x^2}$

Let $\tan^{-1}x = t$ $\quad [\because \frac{d}{dx}(tan^{-1}x) = \frac{1}{1+x^2}]$

Differentiating with respect to t, we get

$\frac{1}{1+x^2}\frac{dx}{dt} = 1$

$\therefore \frac{1}{1+x^2}dx = dt$ **[1 Mark]**

$\Rightarrow \int \frac{e^{\tan^{-1} x}}{1+x^2} dx$

$= \int e^t dt$

[$\frac{1}{2}$ Mark]

$= e^t + C$ $\quad [\because \int e^{ax} dx = \frac{e^{ax}}{a} + C]$ **[1 Mark]**

$= e^{\tan^{-1} x} + C$ $\quad$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function $\frac{e^{\tan^{-1} x}}{1+x^2} = e^{\tan^{-1} x} + C$

[$\frac{1}{2}$ Mark]

19. Integrate the function $\frac{e^{2x}-1}{e^{2x}+1}$ **[4 Marks]**

Solution:

We need to integrate $\frac{e^{2x}-1}{e^{2x}+1}$

Dividing numerator and denominator by e^x, we obtain

$$\frac{\frac{(e^{2x}-1)}{e^x}}{\frac{(e^{2x}+1)}{e^x}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Let $e^x + e^{-x} = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$(e^x - e^{-x})\frac{dx}{dt} = 1$

$\therefore (e^x - e^{-x})dx = dt$ **[1 Mark]**

$\Rightarrow \int \frac{e^{2x}-1}{e^{2x}+1} dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

$= \int \frac{dt}{t}$

[$\frac{1}{2}$ Mark]

$= \log|t| + C$ $\quad [\because \int \frac{1}{x} dx = \log x + C]$ **[1 Mark]**

$= \log|e^x + e^{-x}| + C$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function $\frac{e^{2x}-1}{e^{2x}+1} = \log|e^x + e^{-x}| + C$

[$\frac{1}{2}$ Mark]

20. Integrate the function $\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$ **[4 Marks]**

Solution:

We need to integrate $\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$

Let $e^{2x} + e^{-2x} = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$(2e^{2x} - 2e^{2x})\frac{dx}{dt} = 1$

$\therefore (2e^{2x} - 2e^{2x})dx = dt$

$\Rightarrow 2(e^{2x} - e^{-2x})dx = dt$ **[1 Mark]**

$\Rightarrow \int \left(\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}\right) dx = \int \frac{dt}{2t}$

$= \frac{1}{2}\int \frac{1}{t} dt$

[$\frac{1}{2}$ Mark]

$= \frac{1}{2}\log|t| + C$ $\quad [\because \int \frac{1}{x} dx = \log x + C]$ **[1 Mark]**

$= \frac{1}{2}\log|e^{2x} + e^{-2x}| + C$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function $\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}} = \frac{1}{2}\log|e^{2x} + e^{-2x}| + C$ **[$\frac{1}{2}$ Mark]**

21. Integrate the function $\tan^2(2x-3)$ **[4 Marks]**

Solution:

We need to integrate $\tan^2(2x-3)$

$= \sec^2(2x-3) - 1$

Let $2x - 3 = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$2\frac{dx}{dt} = 1$

$\therefore 2dx = dt$ **[1 Mark]**

$\Rightarrow \int \tan^2(2x-3)dx = \int[(\sec^2(2x-3)) - 1]\,dx$ $[\because \tan^2(x)dx = (\sec^2(x)) - 1]$

[$\frac{1}{2}$ Mark]

$= \frac{1}{2}\int(\sec^2 t)dt - \int 1dx$

$= \frac{1}{2}\int \sec^2 t\,dt - \int 1dx$

$= \frac{1}{2}\tan t - x + C$ $[\because \int \sec^2 x\ dx = \tan\ x + C]$ **[1 Mark]**

$= \frac{1}{2}\tan(2x-3) - x + C$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function $\tan^2(2x-3) = \frac{1}{2}\tan(2x-3) - x + C$

[$\frac{1}{2}$ Mark]

22. Integrate the function $sec^2(7-4x)$ **[4 Marks]**

Solution:

We need to integrate $sec^2(7-4x)$

Let $7 - 4x = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$-4\frac{dx}{dt} = 1$

$\therefore -4dx = dt$ **[1 Mark]**

$\therefore \int \sec^2(7-4x)dx = \frac{-1}{4}\int \sec^2 t\, dt$

[$\frac{1}{2}$ Mark]

$= \frac{-1}{4}(\tan t) + C$ $[\because \int \sec^2 x\ dx = \tan\ x + C]$ **[1 Mark]**

$= \frac{-1}{4}\tan(7-4x) + C$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function $\sec^2(7-4x) = \frac{-1}{4}\tan(7-4x) + C$

[$\frac{1}{2}$ Mark]

23. Integrate the function $\frac{\sin^{-1} x}{\sqrt{1-x^2}}$ **[4 Marks]**

Solution:

We need to integrate $\frac{\sin^{-1} x}{\sqrt{1-x^2}}$

Let $\sin^{-1} x = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$\frac{1}{\sqrt{1-x^2}}\frac{dx}{dt} = 1$ $[\because \frac{d}{dx}(sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}]$

$\Rightarrow \frac{1}{\sqrt{1-x^2}}dx = dt$ **[1 Mark]**

$\Rightarrow \int \frac{\sin^{-1} x}{\sqrt{1-x^2}}dx = \int t\, dt$

[$\frac{1}{2}$ Mark]

$= \frac{t^2}{2} + C$ $[\because \int x^n dx = \frac{x^{n+1}}{n+1} + C]$ **[1 Mark]**

$= \frac{(\sin^{-1}x)^2}{2} + C$ [substituting t]

[$\frac{1}{2}$ **Mark]**

Hence, integration of the function $\frac{\sin^{-1}x}{\sqrt{1-x^2}} = \frac{(\sin^{-1}x)^2}{2} + C$

[$\frac{1}{2}$ **Mark]**

24. Integrate the function $\frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}$ **[4 Marks]**

Solution:

We need to integrate $\frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}$

$$= \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}$$

$$= \frac{2\cos x - 3\sin x}{2(3\cos x + 2\sin x)}$$

Let $3\cos x + 2\sin x = t$ [$\frac{1}{2}$ **Mark]**

Differentiating with respect to t, we get

$(-3\sin x + 2\cos x)\frac{dx}{dt} = 1$ [$\because \frac{d}{dx}(sin\, x) = \cos x\ and\ \frac{d}{dx}(cos\, x) = -\sin x$]

$\Rightarrow (-3\sin x + 2\cos x)dx = dt$ **[1 Mark]**

$$\int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}dx$$

$= \int \frac{dt}{2t}$

[$\frac{1}{2}$ **Mark]**

$= \frac{1}{2}\int \frac{1}{t}dt$

$= \frac{1}{2}\log|t| + C$ [$\because \int \frac{1}{x}dx = \log x + C$] **[1 Mark]**

$\frac{1}{2}\log|2\sin x + 3\cos x| + C$ [substituting t]

[$\frac{1}{2}$ **Mark]**

Hence, integration of the function $\frac{2\cos x-3\sin x}{6\cos x+4\sin x}=\frac{1}{2}\log|2\sin x+3\cos x|+C.$

[$\frac{1}{2}$ Mark]

25. Integrate the function $\frac{1}{\cos^2 x(1-\tan x)^2}$ **[4 Marks]**

Solution:

We need to integrate $\frac{1}{\cos^2 x(1-\tan x)^2}$

$=\frac{\sec^2 x}{(1-\tan x)^2}$ $\quad [\because \frac{1}{\cos^2 x}=\sec^2 x]$

Let $(1-\tan x)=t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$-\sec^2 x\frac{dx}{dt}=1$ $\quad [\because \frac{d}{dx}\tan x=\sec^2 x]$

$-\sec^2 x\, dx=dt$ **[1 Mark]**

$\Rightarrow \int\frac{\sec^2 x}{(1-\tan x)^2}dx=\int\frac{-dt}{t^2}$ **[$\frac{1}{2}$ Mark]**

$=-\int t^{-2}dt$

$=+\frac{1}{t}+C$ $\quad [\because \int x^n dx=\frac{x^{n+1}}{n+1}+C]$ **[1 Mark]**

$=\frac{1}{1-\tan x}+C$ $\quad$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function $\frac{1}{\cos^2 x(1-\tan x)^2}=\frac{1}{1-\tan x}+C$ **[$\frac{1}{2}$ Mark]**

26. Integrate the function $\frac{\cos\sqrt{x}}{\sqrt{x}}$ **[4 Marks]**

Solution:

We need to integrate $\frac{\cos\sqrt{x}}{\sqrt{x}}$

Let $\sqrt{x} = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$\frac{1}{2\sqrt{x}}dx = dt$ [$\because \int x^n dx = \frac{x^{n+1}}{n+1} + C$] **[1 Mark]**

$\Rightarrow \int \frac{\cos\sqrt{x}}{\sqrt{x}}dx = 2\int \cos t dt$

[$\frac{1}{2}$ Mark]

$= 2\sin t + C$ [$\because \int \cos x \, dx = \sin x + C$] **[1 Mark]**

$= 2\sin\sqrt{x} + C$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function $\frac{\cos\sqrt{x}}{\sqrt{x}} = 2\sin\sqrt{x} + C$

[$\frac{1}{2}$ Mark]

27. Integrate the function $\sqrt{\sin 2x}\,\cos 2x$ **[4 Marks]**

Solution:

We need to integrate $\sqrt{\sin 2x}\,\cos 2x$

Let $\sin 2x = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

So, $2\cos 2x\, dx = dt$ [$\because \frac{d}{dx}(\sin x) = \cos x$] **[1 Mark]**

$\Rightarrow \int \sqrt{\sin 2x}\,\cos 2x\, dx = \frac{1}{2}\int \sqrt{t} dt$

[$\frac{1}{2}$ Mark]

$= \frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$ [$\because \int x^n dx = \frac{x^{n+1}}{n+1} + C$] **[1 Mark]**

$= \frac{1}{3}t^{\frac{3}{2}} + C$

$= \frac{1}{3}(\sin 2x)^{\frac{3}{2}} + C$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function $\sqrt{\sin 2x}\cos 2x = \frac{1}{3}(\sin 2x)^{\frac{3}{2}} + C$

[$\frac{1}{2}$ Mark]

28. Integrate the function $\frac{\cos x}{\sqrt{1+\sin x}}$ **[4 Marks]**

Solution:

We need to integrate $\frac{\cos x}{\sqrt{1+\sin x}}$

Let $1 + \sin x = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$\therefore \cos x dx = dt$ $[\because \frac{d}{dx}(sin\, x) = \cos x]$ **[1 Mark]**

$\Rightarrow \int \frac{\cos x}{\sqrt{1+\sin x}} dx$

$= \int \frac{dt}{\sqrt{t}}$

[$\frac{1}{2}$ Mark]

$= \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$ $[\because \int x^n dx = \frac{x^{n+1}}{n+1} + C]$ **[1 Mark]**

$= 2\sqrt{t} + C$

$= 2\sqrt{1 + \sin x} + C$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function $\frac{\cos x}{\sqrt{1+\sin x}} = 2\sqrt{1 + \sin x} + C$

[$\frac{1}{2}$ Mark]

29. Integrate the function $\cot x \log \sin x$ **[4 Marks]**

Solution:

We need to integrate $\cot x \log \sin x$

Let $\log \sin x = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$\Rightarrow \frac{1}{\sin x} \cdot \cos x \, dx = dt$ [$\because \frac{d}{dx}(\log x) = \frac{1}{x}$]

$\therefore \cot x \, dx = dt$ [$\because \frac{\cos x}{\sin x} = \cot x$] **[1 Mark]**

$\Rightarrow \int \cot x \log \sin x \, dx$

$= \int t \, dt$

[$\frac{1}{2}$ Mark]

$= \frac{t^2}{2} + C$ [$\because \int x^n dx = \frac{x^{n+1}}{n+1} + C$] **[1 Mark]**

$= \frac{1}{2}(\log \sin x)^2 + C$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function $\cot x \log \sin x = \frac{1}{2}(\log \sin x)^2 + C$

[$\frac{1}{2}$ Mark]

30. Integrate the function $\frac{\sin x}{1+\cos x}$ **[4 Marks]**

Solution:

We need to integrate $\frac{\sin x}{1+\cos x}$

Let $1 + \cos x = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$\therefore -\sin x\, dx = dt$ $\quad [\because \frac{d}{dx}(\cos x) = -\sin x]$ **[1 Mark]**

$\Rightarrow \int \frac{\sin x}{1+\cos x} dx = \int -\frac{dt}{t}$

[$\frac{1}{2}$ Mark]

$= -\log|t| + C$ $\quad [\because \int \frac{1}{x} dx = \log x + C]$ **[1 Mark]**

$= -\log|1 + \cos x| + C$ $\quad$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function $\frac{\sin x}{1+\cos x} = -\log|1 + \cos x| + C$

[$\frac{1}{2}$ Mark]

31. Integrate the function $\frac{\sin x}{(1+\cos x)^2}$ **[4 Marks]**

Solution:

We need to integrate $\frac{\sin x}{(1+\cos x)^2}$

Let $1 + \cos x = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$-\sin x\, dx = dt$

$dx = \frac{-1}{\sin x} dt$ **[1 Mark]**

$$\therefore \int \frac{\sin x}{(1 + \cos x)^2} dx$$

$$= \int -\frac{dt}{t^2}$$

$= -\int t^{-2} dt$

[$\frac{1}{2}$ Mark]

$= \frac{1}{t} + C$ $\quad [\because \int x^n dx = \frac{x^{n+1}}{n+1} + C]$ **[1 Mark]**

$= \frac{1}{1+\cos x} + C$ $\quad$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function $\frac{\sin x}{(1+\cos x)^2} = \frac{1}{1+\cos x} + C$

[$\frac{1}{2}$ Mark]

32. Integrate the function $\frac{1}{1+\cot x}$ **[6 Marks]**

Solution:

We need to integrate $\frac{1}{1+\cot x}$

Let $I = \int \frac{1}{1+\cot x} dx$

$= \int \frac{1}{1+\frac{\cos x}{\sin x}} dx$ $[\because \frac{\cos x}{\sin x} = \cot x]$

[$\frac{1}{2}$ Mark]

$= \int \frac{\sin x}{\sin x + \cos x} dx$

$= \frac{1}{2} \int \frac{2 \sin x}{\sin x + \cos x} dx$ [$\because$ multiplying and dividing by 2]

$= \frac{1}{2} \int \frac{(\sin x + \cos x) + (\sin x - \cos x)}{(\sin x + \cos x)} dx$ [$\because$ adding and subtracting by $\cos x$]

[1 Mark]

$= \frac{1}{2} \int 1 dx + \frac{1}{2} \int \frac{\sin x - \cos x}{\sin x + \cos x} dx$

$= \frac{1}{2}(x) + \frac{1}{2} \int \frac{\sin x - \cos x}{\sin x + \cos x} dx$

[$\frac{1}{2}$ Mark]

Let $\sin x + \cos x = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$\Rightarrow (\cos x - \sin x)dx = dt$

[1 Mark]

Hence, $I = \frac{x}{2} + \frac{1}{2} \int \frac{(\sin x - \cos x)}{(\cos x + \sin x)(\cos x - \sin x)} (dt)$

$\therefore I = \frac{x}{2} + \frac{1}{2} \int \frac{-(dt)}{t}$

[$\frac{1}{2}$ Mark]

$= \frac{x}{2} - \frac{1}{2}\log|t| + C$ [$\because \int \frac{1}{x} dx = \log x + C$]

[1 Mark]

$= \frac{x}{2} - \frac{1}{2}\log|\sin x + \cos x| + C$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function $\frac{1}{1+\cot x} = \frac{x}{2} - \frac{1}{2}\log|\sin x + \cos x| + C$

[$\frac{1}{2}$ Mark]

33. Integrate the function $\frac{1}{1-\tan x}$ **[6 Marks]**

Solution:

We need to integrate $\frac{1}{1-\tan x}$

Let $I = \int \frac{1}{1-\tan x} dx$

$= \int \frac{1}{1-\frac{\sin x}{\cos x}} dx$ [$\because \tan x = \frac{\sin x}{\cos x}$]

[$\frac{1}{2}$ Mark]

$$= \int \frac{\cos x}{\cos x - \sin x} dx$$

$= \frac{1}{2}\int \frac{2\cos x}{\cos x - \sin x} dx$ [$\because$ multiplying and dividing by 2]

$= \frac{1}{2}\int \frac{(\cos x - \sin x)+(\cos x + \sin x)}{(\cos x - \sin x)} dx$ [$\because$ adding and subtracting by $\sin x$]

[1 Mark]

$$= \frac{1}{2}\int 1 dx + \frac{1}{2}\int \frac{\cos x + \sin x}{\cos x + \sin x} dx$$

$= \frac{x}{2} + \frac{1}{2}\int \frac{\cos x + \sin x}{\cos x - \sin x} dx$

[$\frac{1}{2}$ Mark]

Put $\cos x - \sin x = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$\Rightarrow (-\sin x - \cos x)dx = dt$ $[\because \frac{d}{dx}(sin\ x) = \cos x\ \ and\ \frac{d}{dx}(cos\ x) = -\sin x]$

[1 Mark]

$\therefore I = \frac{x}{2} + \frac{1}{2}\int \frac{-(dt)}{t}$

[$\frac{1}{2}$ Mark]

$= \frac{x}{2} - \frac{1}{2}\log|t| + C$ $[\because \int \frac{1}{x}dx = \log x + C]$

[1 Mark]

$= \frac{x}{2} - \frac{1}{2}\log|\cos x - \sin x| + C$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function $\frac{1}{1-\tan x} = \frac{x}{2} - \frac{1}{2}\log|\cos x - \sin x| + C$

[$\frac{1}{2}$ Mark]

34. Integrate the function $\frac{\sqrt{\tan x}}{\sin x \cos x}$ **[6 Marks]**

Solution:

We need to integrate $\frac{\sqrt{\tan x}}{\sin x \cos x}$

Let $I = \int \frac{\sqrt{\tan x}}{\sin x \cos x}dx$

$= \frac{\int \sqrt{\tan x} \times \cos x}{\sin x \cos x \times \cos x}dx$ [$\because$ multiplying and dividing by $\cos x$] **[1 Mark]**

$= \int \frac{\sqrt{\tan x}}{\tan x \cos^2 x}dx$ $[\because \frac{\sin x}{\cos x} = \tan x]$

$= \int \frac{\sec^2 x dx}{\sqrt{\tan x}}$ $[\because \frac{1}{\cos^2 x} = \sec^2]$ **[1 Mark]**

Let $\tan x = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$\Rightarrow \sec^2 x dx = dt$ **[1 Mark]**

$\therefore I = \int \frac{dt}{\sqrt{t}}$

[$\frac{1}{2}$ Mark]

$= 2\sqrt{t} + C$ [$\because \int x^n dx = \frac{x^{n+1}}{n+1} + C$] **[1 Mark]**

$= 2\sqrt{\tan x} + C$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function $\frac{\sqrt{\tan x}}{\sin x \cos x} = 2\sqrt{\tan x} + C$

[$\frac{1}{2}$ Mark]

35. Integrate the function $\frac{(1+\log x)^2}{x}$ **[4 Marks]**

Solution:

We need to integrate $\frac{(1+\log x)^2}{x}$

Let, $1 + \log x = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$\frac{1}{x}dx = dt$ $\left[\because \frac{d(\log x)}{dx} = \frac{1}{x}\right]$

$\therefore \frac{1}{x}dx = dt$ **[1 Mark]**

$\Rightarrow \int \frac{(1+\log x)^2}{x}dx = \int t^2 dt$

[$\frac{1}{2}$ Mark]

$= \frac{t^3}{3} + C$ [$\because \int x^n dx = \frac{x^{n+1}}{n+1} + C$] **[1 Mark]**

$= \frac{(1+\log x)^3}{3} + C$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function $\frac{(1+\log x)^2}{x} = \frac{(1+\log x)^3}{3} + C$

[$\frac{1}{2}$ Mark]

36. Integrate the function $\frac{(x+1)(x+\log x)^2}{x}$ **[4 Marks]**

Solution:

We need to integrate $\frac{(x+1)(x+\log x)^2}{x}$

$= \left(\frac{x+1}{x}\right)(x+\log x)^2 = \left(1+\frac{1}{x}\right)(x+\log x)^2$

Let, $x + \log x = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$\therefore \left(1+\frac{1}{x}\right)dx = dt$ $\left[\because \frac{d\,(\log x)}{dx} = \frac{1}{x}\right]$ **[1 Mark]**

$\Rightarrow \int \left(1+\frac{1}{x}\right)(x+\log x)^2 dx$

$= \int t^2 dt$

[$\frac{1}{2}$ Mark]

$= \frac{t^3}{3} + C$ **[1 Mark]**

$= \frac{1}{3}(x+\log x)^3 + C$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function $\frac{(x+1)(x+\log x)^2}{x} = \frac{1}{3}(x+\log x)^3 + C$

[$\frac{1}{2}$ Mark]

37. Integrate the function $\frac{x^3 sin(\tan^{-1} x^4)}{1+x^8}$ **[6 Marks]**

Solution:

We need to integrate $\frac{x^3 sin(\tan^{-1} x^4)}{1+x^8}$

Let $x^4 = t$ $[\because \frac{d}{dx}(x^n) = n\,x^{n-1}]$ **[$\frac{1}{2}$ Mark]**

Differentiating with respect to t, we get

$\therefore 4x^3dx = dt$ **[1 Mark]**

$\Rightarrow \int \frac{x^3(\tan^{-1}x^4)}{1+x^8}dx = \frac{1}{4}\int \frac{\sin(\tan^{-1}t)}{1+t^2}dt \ldots(1)$ **[1 Mark]**

Let $\tan^{-1}t = u$

[$\frac{1}{2}$ Mark]

$\therefore \frac{1}{1+t^2}dt = du$ $[\because \frac{d}{dx}(\tan^{-1}) = \frac{1}{1+x^2}]$

[$\frac{1}{2}$ Mark]

From (1), we obtain

$$\int \frac{x^3\sin(\tan^{-1}x^4)dx}{1+x^8} = \frac{1}{4}\int \sin u du$$

$= \frac{1}{4}(-\cos u) + C$ $[\because \int \sin x\,dx = -\cos x + C]$ **[1 Mark]**

$= \frac{-1}{4}\cos(\tan^{-1}t) + C$ [substituting u]

[$\frac{1}{2}$ Mark]

$= \frac{-1}{4}\cos(\tan^{-1}x^4) + C$ [substituting t]

[$\frac{1}{2}$ Mark]

Hence, integration of the function $\frac{x^3 sin(\tan^{-1}x^4)}{1+x^8} = \frac{-1}{4}\cos(\tan^{-1}x^4) + C$

[$\frac{1}{2}$ Mark]

38. $\int \frac{10x^9+10^x\log_e 10}{x^{10}+10^x}dx$ equals **[4 Marks]**

(A) $10^x - x^{10} + C$

(B) $10^x + x^{10} + C$

(C) $(10^x - x^{10})^{-1} + C$

(D) $\log(10^x + x^{10}) + C$

Solution:

Let $x^{10} + 10^x = t$

[$\frac{1}{2}$ Mark]

Differentiating with respect to t, we get

$\therefore (10x^9 + 10^x \log_e 10)dx = dt$ $[\because \frac{d}{dx}(x^n) = n\,x^{n-1}]$

[1 Mark]

$\Rightarrow \int \frac{10x^9+10^x \log_e 10}{x^{10}+10^x} dx$

$= \int \frac{dt}{t}$

[$\frac{1}{2}$ Mark]

$= \log t + C$ $[\because \int \frac{1}{x} dx = \log x + C]$

[1 Mark]

$= \log(10^x + x^{10}) + C$

[$\frac{1}{2}$ Mark]

Hence, the correct Answer is D **[$\frac{1}{2}$ Mark]**

39. $\int \frac{dx}{\sin^2 x \cos^2 x}$ equals **[4 Marks]**

(A) $\tan x + \cot x + C$

(B) $\tan x - \cot x + C$

(C) $\tan x \cot x + C$

(D) $\tan x - \cot 2x + C$

Solution:

Let $I = \int \frac{dx}{\sin^2 x \cos^2 x}$

$$= \int \frac{1}{\sin^2 x \cos^2 x} dx$$

$= \int \frac{\sin^2 x+\cos^2 x}{\sin^2 x \cos^2 x}$ **[1 Mark]**

$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx$ **[$\frac{1}{2}$ Mark]**

$= \int \sec^2 x\, dx + \int cosec^2 x dx$ $[\because \left(\frac{1}{\cos^2 x}\right) = \sec^2 x$ *and* $\left(\frac{1}{\sin^2 x}\right) = \operatorname{cosec}^2 x]$

[1 Mark]

$= \tan x - \cot x + C$ $[\because \int \sec^2 x\, dx = \tan x + c$ *and* $\int \operatorname{cosec}^2 x\, dx = -\cot x + C]$

[1 Mark]

Hence, the correct Answer is B. **[$\frac{1}{2}$ Mark]**

Exercise 7.3

1. Find the integral of the function $\sin^2(2x+5)$ **[4 Marks]**

Solution:

We need to integrate $\sin^2(2x+5)$

$=\frac{1-\cos 2(2x+5)}{2}=\frac{1-\cos(4x+10)}{2}$ $\quad [\because \sin^2(x)=\frac{1-\cos(2x)}{2}]$ **[$\frac{1}{2}$ Mark]**

$\Rightarrow \int \sin^2(2x+5)dx$

$=\int \frac{1-\cos(4x+10)}{2}dx$ **[$\frac{1}{2}$ Mark]**

$=\frac{1}{2}\int 1\,dx-\frac{1}{2}\int \cos(4x+10)dx$

$=\frac{1}{2}x-\frac{1}{2}\left(\frac{\sin(4x+10)}{4}\right)+C$ $\quad [\because \int \cos x\,\mathrm{dx}=\sin x+C]$ **[2 Marks]**

$=\frac{1}{2}x-\frac{1}{8}\sin(4x+10)+C$

Hence, integration of the function $\sin^2(2x+5)=\frac{1}{2}x-\frac{1}{8}\sin(4x+10)+C$ **[1 Mark]**

2. Find the integral of the function $\sin 3x\cos 4x$ **[4 Marks]**

Solution:

We need to integrate $\sin 3x\cos 4x$

It is known that, $\sin A\cos B=\frac{1}{2}\{\sin(A+B)+\sin(A-B)\}$

$\therefore \int \sin 3x \cos 4x\, dx = \frac{1}{2}\int\{\sin(3x+4x) + \sin(3x-4x)\}dx$ **[1 Mark]**

$= \frac{1}{2}\int\{\sin 7x + \sin(-x)\}dx$

$= \frac{1}{2}\int\{\sin 7x - \sin x\}dx$

$= \frac{1}{2}\int \sin 7x\, dx - \frac{1}{2}\int \sin x\, dx$

$= \frac{1}{2}\left(\frac{-\cos 7x}{7}\right) - \frac{1}{2}(-\cos x) + C$ $\quad [\because \int \sin ax\, \mathrm{dx} = \frac{-\cos ax}{a} + C]$ **[2 Marks]**

$= \frac{-\cos 7x}{14} + \frac{\cos x}{2} + C$

Hence, integration of the function $\sin 3x \cos 4x = \frac{-\cos 7x}{14} + \frac{\cos x}{2} + C$ **[1 Mark]**

3. Find the integral of the function $\cos 2x \cos 4x \cos 6x$ **[4 Marks]**

Solution:

We need to integrate $\cos 2x \cos 4x \cos 6x$

It is known that, $\cos A \cos B = \frac{1}{2}\{\cos(A+B) + \cos(A-B)\}$

$\therefore \int \cos 2x(\cos 4x \cos 6x)dx$

$= \int \cos 2x\left[\frac{1}{2}\{\cos(4x+6x) + \cos(4x-6x)\}\right]dx$ **[1 Mark]**

$= \frac{1}{2}\int\{\cos 2x \cos 10x + \cos 2x \cos(-2x)\}dx$

$= \frac{1}{2}\int\{\cos 2x \cos 10x + \cos 2x \cos^2 2x\}dx$

$= \frac{1}{2}\int\left[\left\{\frac{1}{2}\cos(2x+10x) + \cos(2x-10x)\right\} + \left(\frac{1+\cos 4x}{2}\right)\right]dx$ **[$\frac{1}{2}$ Mark]**

$= \frac{1}{4}\left[\frac{\sin 12x}{12} + \frac{\sin 8x}{8} + x + \frac{\sin 4x}{4}\right] + C$ [$\because \int \sin ax\, \mathrm{dx} = \frac{-\cos ax}{a} + C$] **[2 Marks]**

Hence, integration of the function $\cos 2x \cos 4x \cos 6x = \frac{1}{4}\left[\frac{\sin 12x}{12} + \frac{\sin 8x}{8} + x + \frac{\sin 4x}{4}\right] + C$

[$\frac{1}{2}$ Mark]

4. Find the integral of the function $\sin^3(2x+1)$ **[6 Marks]**

Solution:

We need to integrate $\sin^3(2x+1)$

Let $I = \int \sin^3(2x+1)$

$\Rightarrow \int \sin^3(2x+1)\, dx = \int \sin^2(2x+1) \cdot \sin(2x+1) dx$

$= \int (1 - \cos^2(2x+1)) \sin(2x+1) dx$ [$\because \sin^2(x) + \cos^2(x) = 1$] **[1 Mark]**

Let $\cos(2x+1) = t$ **[$\frac{1}{2}$ Mark]**

Differentiating with respect to t,

$\Rightarrow -2\sin(2x+1) dx = dt$ [$\because \frac{d}{dx}(\cos x) = -\sin x$] **[1 Mark]**

$\Rightarrow \sin(2x+1) dx = \frac{-dt}{2}$

Now, $I = \int (1 - \cos^2(2x+1)) \sin(2x+1) dx$

$\Rightarrow I = \frac{-1}{2} \int (1 - t^2) dt$ **[$\frac{1}{2}$ Mark]**

$= \frac{-1}{2}\left\{t - \frac{t^3}{3}\right\} + C$ [$\because \int x^n dx = \frac{x^{n+1}}{n+1} + C$] **[2 Marks]**

$= \frac{-1}{2}\left\{\cos(2x+1) - \frac{\cos^3(2x+1)}{3}\right\} + C$ [$\because$ Substituting t]

$= \frac{-\cos(2x+1)}{2} + \frac{\cos^3(2x+1)}{6} + C$
[$\frac{1}{2}$ Mark]

Hence, integration of the function $\sin^3(2x+1) = \frac{-\cos(2x+1)}{2} + \frac{\cos^3(2x+1)}{6} + C$ **[$\frac{1}{2}$ Mark]**

5. Find the integral of the function $\sin^3 x \cos^3 x$ **[6 Marks]**

Solution:

We need to integrate $\sin^3 x \cos^3 x$

Let $I = \int \sin^3 x \cos^3 x \, dx$

$= \int \cos^3 x \cdot \sin^2 x \cdot \sin x \cdot dx$

$= \int \cos^3 x(1 - \cos^2 x) \sin x \cdot dx$ $[\because \sin^2(x) + \cos^2(x) = 1]$ **[1 Mark]**

Let $\cos x = t$ **[$\frac{1}{2}$ Mark]**

Differentiating with respect to t,

$\Rightarrow -\sin x \cdot dx = dt$ **[1 Mark]**

$\Rightarrow I = \int t^3(1 - t^2)dt$ **[$\frac{1}{2}$ Mark]**

$= -\int (t^3 - t^5)dt$

$= -\left\{\frac{t^4}{4} - \frac{t^6}{6}\right\} + C$ **[2 Marks]**

$= -\left\{\frac{\cos^4 x}{4} - \frac{\cos^6 x}{6}\right\} + C$ [$\because$ Substituting t]

$= \frac{\cos^6 x}{6} - \frac{\cos^4 x}{4} + C$
[$\frac{1}{2}$ Mark]

Hence, integration of the function $\sin^3 x \cos^3 x = \frac{\cos^6 x}{6} - \frac{\cos^4 x}{4} + C$ **[$\frac{1}{2}$ Mark]**

6. Find the integral of the function $\sin x \sin 2x \sin 3x$ **[6 Marks]**

Solution:

We need to integrate $\sin x \sin 2x \sin 3x$

We know that, $\sin A \sin B = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\}$

$$\therefore \int \sin x \sin 2x \sin 3x \, dx$$

$= \int \left[\sin x \cdot \frac{1}{2}\{\cos(2x-3x) - \cos(2x+3x)\}\right] dx$ **[1 Mark]**

$= \frac{1}{2}\int (\sin x \cos(-x) - \sin x \cos 5x) dx$

$= \frac{1}{2}\int (\sin x \cos x - \sin x \cos 5x) dx$

$= \frac{1}{2}\int \frac{\sin 2x}{2} dx - \frac{1}{2}\int \sin x \cos 5x \, dx$ $[\because 2\sin x \cos x = \sin 2x]$

$= \frac{1}{4}\left[\frac{-\cos 2x}{2}\right] - \frac{1}{2}\int \left\{\frac{1}{2}\sin(x+5x) + \sin(x-5x)\right\} dx$ **[2 Marks]**

$[\because \sin A \cos B = \frac{1}{2}\{\sin(A+B) + \sin(A-B)\}]$

$= \frac{1}{4}\left[\frac{-\cos 2x}{2}\right] - \frac{1}{2}\int \left\{\frac{1}{2}\sin(6x) - \sin(4x)\right\} dx$

$= \frac{-\cos 2x}{8} - \frac{1}{4}\left[\frac{-\cos 6x}{3} + \frac{\cos 4x}{4}\right] + C$ $[\because \int \sin ax \, dx = \frac{-\cos ax}{a} + C]$ **[2 Marks]**

$= \frac{-\cos 2x}{8} - \frac{1}{8}\left[\frac{-\cos 6x}{3} + \frac{\cos 4x}{2}\right] + C$

$= \frac{1}{8}\left[\frac{\cos 6x}{3} - \frac{\cos 4x}{2} - \cos 2x\right] + C$ **[$\frac{1}{2}$ Mark]**

Hence, integration of the function $\sin x \sin 2x \sin 3x = \frac{1}{8}\left[\frac{\cos 6x}{3} - \frac{\cos 4x}{2} - \cos 2x\right] + C$ **[$\frac{1}{2}$ Mark]**

7. Find the integral of the function $\sin 4x \sin 8x$ **[4 Marks]**

Solution:

We need to integrate $\sin 4x \sin 8x$

It is known that, $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$\therefore \int \sin 4x \sin 8x \, dx = \int \left\{\frac{1}{2}[\cos(4x-8x) - \cos(4x+8x)]\right\} dx$ **[1 Mark]**

$= \frac{1}{2}\int (\cos(-4x) - \cos 12x) dx$

$= \frac{1}{2}\int (\cos 4x - \cos 12x) dx$

$= \frac{1}{2}\left[\frac{\sin 4x}{4} - \frac{\sin 12x}{12}\right] + C$ $\quad [\because \int \sin ax \, dx = \frac{-\cos ax}{a} + C]$ **[2 Marks]**

Hence, integration of the function $\sin 4x \sin 8x = \frac{1}{2}\left[\frac{\sin 4x}{4} - \frac{\sin 12x}{12}\right] + C$ **[1 Mark]**

8. Find the integral of the function $\frac{1-\cos x}{1+\cos x}$ **[2 Marks]**

Solution:

We need to integrate $\frac{1-\cos x}{1+\cos x}$

$\frac{1-\cos x}{1+\cos x} = \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$ $\left[2\sin^2\frac{x}{2} = 1 - \cos x \text{ and } 2\cos^2\frac{x}{2} = 1 + \cos x\right]$

$= \tan^2\frac{x}{2}$

$= \left(\sec^2\frac{x}{2} - 1\right)$ **[$\frac{1}{2}$ Mark]**

$\therefore \int \frac{1-\cos x}{1+\cos x} dx = \int \left(\sec^2\frac{x}{2} - 1\right) dx$

$$= \left[\frac{\tan\frac{x}{2}}{\frac{1}{2}} - x\right] + C$$

$= 2\tan\frac{x}{2} - x + C$ **[1 Mark]**

Hence, integration of the function $\frac{1-\cos x}{1+\cos x} = 2\tan\frac{x}{2} - x + C$ **[$\frac{1}{2}$ Mark]**

9. Find the integral of the function $\frac{\cos x}{1+\cos x}$ **[4 Marks]**

Solution:

We need to integrate $\frac{\cos x}{1+\cos x}$

$\frac{\cos x}{1+\cos x} = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$ $\left[\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} \text{ and } \cos x = 2\cos^2\frac{x}{2} - 1\right]$

$= \frac{1}{2}\left[1 - \tan^2\frac{x}{2}\right]$ **[1 Mark]**

$\therefore \int \frac{\cos x}{1+\cos x}dx = \frac{1}{2}\int\left(1 - \tan^2\frac{x}{2}\right)dx$

$= \frac{1}{2}\int\left(1 - \sec^2\frac{x}{2} + 1\right)dx$

$= \frac{1}{2}\int\left(2 - \sec^2\frac{x}{2}\right)dx$

$= \frac{1}{2}\left[2x - \frac{\tan\frac{x}{2}}{\frac{1}{2}} + C\right]$

$= x - \tan\frac{x}{2} + C$ **[2 Marks]**

Hence, integration of the function $\frac{\cos x}{1+\cos x} = x - \tan\frac{x}{2} + C$ **[1 Mark]**

10. Find the integral of the function $\sin^4 x$ **[4 Marks]**

Solution:

We need to integrate $\sin^4 x$

$\sin^4 x = \sin^2 x \sin^2 x$

$= \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1-\cos 2x}{2}\right)$

$= \frac{1}{4}(1-\cos 2x)^2$

$= \frac{1}{4}[1+\cos^2 2x - 2\cos 2x]$

$= \frac{1}{4}\left[1+\left(\frac{1+\cos 4x}{2}\right) - 2\cos 2x\right]$

$= \frac{1}{4}\left[1+\frac{1}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right]$

$= \frac{1}{4}\left[\frac{3}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right]$ **[2 Marks]**

$\therefore \int \sin^4 x\, dx = \frac{1}{4}\int \left[\frac{3}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right] dx$

$= \frac{1}{4}\left[\frac{3}{2}x+\frac{1}{2}\left(\frac{\sin 4x}{4}\right) - \frac{2\sin 2x}{2}\right] + C$

$= \frac{1}{8}\left[3x+\frac{\sin 4x}{4} - 2\sin 2x\right] + C$

$= \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$

Hence, integration of the function $\sin^4 x = \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$ **[1 Mark]**

11. Find the integral of the function $\cos^4 2x$ **[4 Marks]**

Solution:

We need to integrate $\cos^4 2x$

$\cos^4 2x = (\cos^2 2x)^2$

$= \left(\frac{1+\cos 4x}{2}\right)^2$

$= \frac{1}{4}[1 + \cos^2 4x + 2\cos 4x]$

$= \frac{1}{4}\left[1 + \left(\frac{1 + \cos 8x}{2}\right) + 2\cos 4x\right]$

$= \frac{1}{4}\left[1 + \frac{1}{2} + \frac{\cos 8x}{2} + 2\cos 4x\right]$

$= \frac{1}{4}\left[\frac{3}{2} + \frac{\cos 8x}{2} + 2\cos 4x\right]$ **[2 Marks]**

$\therefore \int \cos^4 2x\, dx = \int \left(\frac{3}{8} + \frac{\cos 8x}{8} + \frac{\cos 4x}{2}\right) dx$

$= \frac{3}{8}x + \frac{\sin 8x}{64} + \frac{\sin 4x}{8} + C$

Hence, integration of the function $\cos^4 2x = \frac{3}{8}x + \frac{\sin 8x}{64} + \frac{\sin 4x}{8} + C$ **[2 Marks]**

12. Find the integral of the function $\frac{\sin^2 x}{1+\cos x}$ **[4 Marks]**

Solution:

We need to integrate $\frac{\sin^2 x}{1+\cos x}$

$\frac{\sin^2 x}{1+\cos x} = \frac{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}}$ $\left[\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}; \cos x = 2\cos^2\frac{x}{2} - 1\right]$

$= \frac{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$

$= 2\sin^2\frac{x}{2}$

$= 1 - \cos x$ **[2 Marks]**

$\therefore \int \frac{\sin^2 x}{1 + \cos x} dx = \int (1 - \cos x) dx$

$= x - \sin x + C$

Hence, integration of the function $\frac{\sin^2 x}{1+\cos x} = x - \sin x + C$ [2 Marks]

13. Find the integral of the function $\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$ [4 Marks]

Solution:

We need to integrate $\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$

$\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} = \frac{-2\sin\frac{2x+2\alpha}{2}\sin\frac{2x-2\alpha}{2}}{-2\sin\frac{x+\alpha}{2}\sin\frac{x-\alpha}{2}}$ $\left[\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}\right]$ [1 Mark]

$$= \frac{\sin(x+\alpha)\sin(x-\alpha)}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$= \frac{\left[2\sin\left(\frac{x+\alpha}{2}\right)\right]\left[2\sin\left(\frac{x-\alpha}{2}\right)\ \cos\left(\frac{x-\alpha}{2}\right)\right]}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$= 4\cos\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)$$

$$= 2\left[\cos\left(\frac{x+\alpha}{2}+\frac{x-\alpha}{2}\right)+\cos\left(\frac{x+\alpha}{2}-\frac{x-\alpha}{2}\right)\right]$$

$$= 2[\cos(x) + \cos\alpha]$$

$$= 2\cos x + 2\cos\alpha$$ [2 Marks]

$$\therefore \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}dx = \int (2\cos x + 2\cos\alpha)\,dx$$

$$= [2\sin x + 2x\cos\alpha] + C$$

$$= 2[\sin x + x\cos\alpha] + C$$

Hence, integration of the function $\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} = 2[\sin x + x\cos\alpha] + C$ [1 Mark]

14. Find the integral of the function $\frac{\cos x-\sin x}{1+\sin 2x}$ **[4 Marks]**

Solution:

We need to integrate $\frac{\cos x-\sin x}{1+\sin 2x}$

$\frac{\cos x-\sin x}{1+\sin 2x}=\frac{\cos x-\sin x}{(\sin^2 x+\cos^2 x)+2\sin x\cos x}$

$[\sin^2 x+\cos^2 x=1; \sin 2x=2\sin x\cos x]$

$=\frac{\cos x-\sin x}{(\sin x+\cos x)^2}$ **[1 Mark]**

Let $\sin x+\cos x=t$ **[$\frac{1}{2}$ Mark]**

$\therefore (\cos x-\sin x)dx=dt$ **[1 Mark]**

$$\Rightarrow \int \frac{\cos x-\sin x}{1+\sin 2x}dx=\int \frac{\cos x-\sin x}{(\sin x+\cos x)^2}dx$$

$=\int \frac{dt}{t^2}$ **[$\frac{1}{2}$ Mark]**

$=\int t^{-2}dt$

$=-t^{-1}+C$

$=-\frac{1}{t}+C$

$=\frac{-1}{\sin x+\cos x}+C$

Hence, integration of the function $\frac{\cos x-\sin x}{1+\sin 2x}=\frac{-1}{\sin x+\cos x}+C$ **[1 Mark]**

15. Find the integral of the function $\tan^3 2x\sec 2x$ **[4 Marks]**

Solution:

We need to integrate $\tan^3 2x\sec 2x$

$\tan^3 2x \sec 2x = \tan^2 2x \tan 2x \sec 2x$

$= (\sec^2 2x - 1) \tan 2x \sec 2x$

$= \sec^2 2x \cdot \tan 2x \sec 2x - \tan 2x \sec 2x$ **[1 Mark]**

$\therefore \int \tan^3 2x \sec 2x \, dx = \int \sec^2 2x \tan 2x \sec 2x \, dx - \int \tan 2x \sec 2x \, dx$

$= \int \sec^2 2x \tan 2x \sec 2x \, dx - \frac{\sec 2x}{2} + C$ **[$\frac{1}{2}$ Mark]**

Let $\sec 2x = t$ **[$\frac{1}{2}$ Mark]**

$\therefore 2 \sec 2x \tan 2x \, dx = dt$

$\therefore \int \tan^3 2x \sec 2x \, dx = \frac{1}{2} \int t^2 dt - \frac{\sec 2x}{2} + C$ **[1 Mark]**

$= \frac{t^3}{6} - \frac{\sec 2x}{2} + C$

$= \frac{(\sec 2x)^3}{6} - \frac{\sec 2x}{2} + C$

Hence, integration of the function $\tan^3 2x \sec 2x = \frac{(\sec 2x)^3}{6} - \frac{\sec 2x}{2} + C$ **[1 Mark]**

16. Find the integral of the function $\tan^4 x$ **[4 Marks]**

Solution:

We need to integrate $\tan^4 x$

$= \tan^2 x \cdot \tan^2 x$

$= (\sec^2 x - 1)\tan^2 x$

$= \sec^2 x \tan^2 x - \tan^2 x$

$= \sec^2 x \tan^2 x - (\sec^2 x - 1)$

$= \sec^2 x \tan^2 x - \sec^2 x + 1$ **[1 Mark]**

$\therefore \int \tan^4 x \, dx = \int \sec^2 x \tan^2 x \, dx - \int \sec^2 x \, dx + \int 1 \cdot dx$

$= \int \sec^2 x \tan^2 x\, dx - \tan x + x + C \ldots (i)$ **[1 Mark]**

Consider $\int \sec^2 x \tan^2 x\, dx$

Let $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$\Rightarrow \int \sec^2 x \tan^2 x dx = \int t^2\, dt = \frac{t^3}{3} = \frac{\tan^3 x}{3}$ **[1 Mark]**

From equation (i), we obtain

$\int \tan^4 x\, dx = \frac{\tan^3 x}{3} - \tan x + x + C$

Hence, integration of the function $\tan^4 x = \frac{1}{3}\tan^3 x - \tan x + x + C$ **[1 Mark]**

17. Find the integral of the function $\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$ **[4 Marks]**

Solution:

We need to integrate $\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$

$\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} = \frac{\sin^3 x}{\sin^2 x \cos^2 x} + \frac{\cos^3 x}{\sin^2 \cos^2 x}$

$= \frac{\sin x}{\cos^2 x} + \frac{\cos x}{\sin^2 x}$

$= \tan x \sec x + \cot x \operatorname{cosec} x$ **[2 Marks]**

$$\therefore \int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} dx = \int (\tan x \sec x + \cot x \operatorname{cosec} x)\, dx$$

$= \sec x - \operatorname{cosec} x + C$

Hence, integration of the function $\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} = \sec x - \operatorname{cosec} x + C$ **[2 Marks]**

18. Find the integral of the function $\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$ **[2 Marks]**

Solution:

We need to integrate $\frac{\cos 2x+2\sin^2 x}{\cos^2 x}$

$=\frac{\cos 2x+(1-\cos 2x)}{\cos^2 x}[\cos 2x=1-2\sin^2 x]$

$=\frac{1}{\cos^2 x}$

$=\sec^2 x$ **[1 Mark]**

Hence, $\int\frac{\cos 2x+2\sin^2 x}{\cos^2 x}dx=\int\sec^2 x\,dx=\tan x+C$ **[1 Mark]**

19. Find the integral of the function $\frac{1}{\sin x\cos^3 x}$ **[4 Marks]**

Solution:

We need to integrate $\frac{1}{\sin x\cos^3 x}$

$=\frac{\sin^2 x+\cos^2 x}{\sin x\cos^3 x}$

$=\frac{\sin x}{\cos^3 x}+\frac{1}{\sin x\cos x}$

$=\tan x\sec^2 x+\frac{\sec^2 x}{\frac{\sin x}{\cos x}\cos^2 x\sec^2 x}$

$=\tan x\sec^2 x+\frac{\sec^2 x}{\tan x}$ **[1 Mark]**

$\therefore\int\frac{1}{\sin x\cos^3 x}dx=\int\tan x\sec^2 x\,dx+\int\frac{\sec^2 x}{\tan x}dx$

Let $\tan x=t\Rightarrow\sec^2 x\,dx=dt$

$\Rightarrow\int\frac{1}{\sin x\cos^3 x}dx=\int t dt+\int\frac{1}{t}dt$ **[1 Mark]**

$=\frac{t^2}{2}+\log|t|+C$

$= \frac{1}{2}\tan^2 x + \log|\tan x| + C$

Hence, integration of the function $\frac{1}{\sin x \cos^3 x} = \frac{1}{2}\tan^2 x + \log|\tan x| + C$ **[2 Marks]**

20. Find the integral of the function $\frac{\cos 2x}{(\cos x+\sin x)^2}$ **[6 Marks]**

Solution:

We need to integrate $\frac{\cos 2x}{(\cos x+\sin x)^2}$

$= \frac{\cos 2x}{\cos^2 x+\sin^2 x+2\sin x\cos x} = \frac{\cos 2x}{1+\sin 2x}$

$\therefore \int \frac{\cos 2x}{(\cos x+\sin x)^2} dx = \int \frac{\cos 2x}{(1+\sin 2x)} dx$ **[2 Marks]**

Let $1 + \sin 2x = t$

$\Rightarrow 2\cos 2x\, dx = dt$ **[1 Mark]**

$\therefore \int \frac{\cos 2x}{(\cos x+\sin x)^2} dx = \int \frac{\cos 2x}{(1+\sin 2x)} dx$

$= \frac{1}{2}\int \frac{1}{t} dt$ **[1 Mark]**

$= \frac{1}{2}\log|t| + C$

$= \frac{1}{2}\log|1 + \sin 2x| + C$

$= \frac{1}{2}\log|(\sin x + \cos x)^2| + C$

$= \log|\sin x + \cos x| + C$

Hence, integration of the function $\frac{\cos 2x}{(\cos x+\sin x)^2} = \log|\sin x + \cos x| + C$ **[2 Marks]**

21. Find the integral of the function $\sin^{-1}(\cos x)$ **[6 Marks]**

Solution:

We need to integrate $\sin^{-1}(\cos x)$

Let $\cos x = t$ Hence, $\sin x = \sqrt{1-t^2}$

$\Rightarrow (-\sin x)dx = dt$

$dx = \dfrac{-dt}{\sin x}$

$dx = \frac{-dt}{\sqrt{1-t^2}}$ **[1 Mark]**

$\therefore \int \sin^{-1}(\cos x)dx = \int \sin^{-1} t \left(\dfrac{-dt}{\sqrt{1-t^2}}\right)$

$= -\int \frac{\sin^{-1} t}{1-t^2} dt$ **[1 Mark]**

Let $\sin^{-1} t = u$

$\Rightarrow \frac{1}{\sqrt{1-t^2}} dt = du$ **[1 Mark]**

$\therefore \int \sin^{-1}(\cos x)dx = -\int u du$

$= -\dfrac{u^2}{2} + C$

$= \dfrac{-(\sin^{-1} t)^2}{2} + C$

$= -\frac{[\sin^{-1}(\cos x)]^2}{2} + C \dots (i)$ **[1 Mark]**

It is known that,

$\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$

$\therefore \sin^{-1}(\cos x) = \frac{\pi}{2} - \cos^{-1}(\cos x) = \left(\frac{\pi}{2} - x\right)$ **[1 Mark]**

Substituting in equation (i), we obtain

$\int \sin^{-1}(\cos x)dx = \frac{-\left[\frac{\pi}{2}-x\right]^2}{2} + C$

$= -\frac{1}{2}\left(\frac{\pi^2}{4} + x^2 - \pi x\right) + C$

$= -\frac{\pi^2}{8} - \frac{x^2}{2} + \frac{1}{2}\pi x + C$

$= \frac{\pi x}{2} - \frac{x^2}{2} + C_1$

Hence, integration of the function $\sin^{-1}(\cos x) = \frac{\pi x}{2} - \frac{x^2}{2} + C_1$ **[1 Mark]**

22. Find the integral of the function $\frac{1}{\cos(x-a)\cos(x-b)}$ **[6 Marks]**

Solution:

We need to integrate $\frac{1}{\cos(x-a)\cos(x-b)}$

$= \frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x-a)\cos(x-b)}\right]$ **[1 Mark]**

$= \frac{1}{\sin(a-b)}\left[\frac{\sin[(x-b)-(x-a)]}{\cos(x-a)\cos(x-b)}\right]$

$= \frac{1}{\sin(a-b)}\frac{[\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)]}{\cos(x-a)\cos(x-b)}$

$= \frac{1}{\sin(a-b)}[\tan(x-b) - \tan(x-a)]$ **[2 Marks]**

$\Rightarrow \int \frac{1}{\cos(x-a)\cos(x-b)}dx = \frac{1}{\sin(a-b)}\int [\tan(x-b) - \tan(x-a)]dx$

$= \frac{1}{\sin(a-b)}[-\log|\cos(x-b)| + \log|\cos(x-a)|] + C$ **[2 Marks]**

$= \frac{1}{\sin(a-b)}\left[\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|\right] + C$

Hence, integration of the function $\frac{1}{\cos(x-a)\cos(x-b)} = \frac{1}{\sin(a-b)}\left[\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|\right] + C$ **[1 Mark]**

23. $\int \frac{\sin^2 x-\cos^2 x}{\sin^2 x \cos^2 x} dx$ is equal to **[2 Marks]**

(A) $\tan x+\cot x+C$

(B) $\tan x+\operatorname{cosec} x+C$

(C) $-\tan x+\cot x+C$

(D) $\tan x+\sec x+C$

Solution:

Given integral is $\int \frac{\sin^2 x-\cos^2 x}{\sin^2 x \cos^2 x} dx$

$=\int\left(\frac{\sin^2 x}{\sin^2 x \cos^2 x}-\frac{\cos^2 x}{\sin^2 x \cos^2 x}\right) dx$

$=\int\left(\sec^2 x-cosec^2 x\right) dx$

$=\tan x+\cot x+C$

Hence, $\int \frac{\sin^2 x-\cos^2 x}{\sin^2 x \cos^2 x} dx=\tan x+\cot x+C$

Hence, the correct Answer is A. **[2 Marks]**

24. $\int \frac{e^x(1+x)}{\cos^2(e^x x)} dx$ equals **[4 Marks]**

(A) $-\cot(ex^x)+C$

(B) $\tan(xe^x)+C$

(C) $\tan(e^x)+C$

(D) $\cot(e^x)+C$

Solution:

Given integral is $\int \frac{e^x(1+x)}{\cos^2(e^x x)} dx$

Let $e^x x=t$

[$\frac{1}{2}$ Mark]

$\Rightarrow (e^x \cdot x + e^x \cdot 1)dx = dt$

$e^x(x+1)dx = dt$ **[1 Mark]**

$\therefore \int \frac{e^x(1+x)}{\cos^2(e^x x)} dx = \int \frac{dt}{\cos^2 t}$ **[$\frac{1}{2}$ Mark]**

$= \int \sec^2 t \, dt$

$= \tan t + C$ **[1 Mark]**

$= \tan(e^x x) + C$ **[$\frac{1}{2}$ Mark]**

Hence, $\int \frac{e^x(1+x)}{\cos^2(e^x x)} dx = \tan(e^x x) + C$

Hence, the correct Answer is B. **[$\frac{1}{2}$ Mark]**

Exercise 7.4

1. Integrate the function $\frac{3x^2}{x^6+1}$ **[2 Marks]**

Solution:

We need to integrate the function $\frac{3x^2}{x^6+1}$

Let $x^3 = t$

$\therefore 3x^2 dx = dt$ **[$\frac{1}{2}$ Mark]**

$$\Rightarrow \int \frac{3x^2}{x^6+1} dx = \int \frac{dt}{t^2+1}$$

$= \tan^{-1} t + C$

$= \tan^{-1}(x^3) + C$ [Substituting t] **[1 Mark]**

Hence, integration of $\frac{3x^2}{x^6+1} = \tan^{-1}(x^3) + C$ **[$\frac{1}{2}$ Mark]**

2. Integrate the function $\frac{1}{\sqrt{1+4x^2}}$ **[2 Marks]**

Solution:

We need to integrate the function $\frac{1}{\sqrt{1+4x^2}}$

Let $2x = t$

$\therefore 2dx = dt$

[$\frac{1}{2}$ Mark]

$$\Rightarrow \int \frac{1}{\sqrt{1+4x^2}} dx = \frac{1}{2} \int \frac{dt}{\sqrt{1+t^2}}$$

$= \frac{1}{2}\left[\log\left|t + \sqrt{t^2+1}\right|\right] + C \quad \left[\int \frac{1}{\sqrt{x^2+a^2}} dx = \log\left|x + \sqrt{x^2+a^2}\right|\right]$

$= \frac{1}{2}\log\left|2x + \sqrt{4x^2+1}\right| + C$ [Substituting t] **[1 Mark]**

Hence, integration of $\frac{1}{\sqrt{1+4x^2}} = \frac{1}{2}\log\left|2x + \sqrt{4x^2+1}\right| + C$

[$\frac{1}{2}$ Mark]

3. Integrate the function $\frac{1}{\sqrt{(2-x)^2+1}}$ **[2 Marks]**

Solution:

We need to integrate the function $\frac{1}{\sqrt{(2-x)^2+1}}$

Let $2 - x = t$

$\Rightarrow -dx = dt$

[$\frac{1}{2}$ Mark]

$\Rightarrow \int \frac{1}{\sqrt{(2-x)^2+1}} dx = -\int \frac{1}{\sqrt{t^2+1}} dt$

$= -\log\left|t + \sqrt{t^2+1}\right| + C$ $\left[\int \frac{1}{\sqrt{x^2+a^2}} dx = \log\left|x + \sqrt{x^2+a^2}\right|\right]$

$= -\log\left|2 - x + \sqrt{(2-x)^2+1}\right| + C$ [Substituting t]

$= \log\left|\frac{1}{(2-x)+\sqrt{x^2-4x+5}}\right| + C$ **[1 Mark]**

Hence, integration of $\frac{1}{\sqrt{(2-x)^2+1}} = \log\left|\frac{1}{(2-x)+\sqrt{x^2-4x+5}}\right| + C$

[$\frac{1}{2}$ Mark]

4. Integrate the function $\frac{1}{\sqrt{9-25x^2}}$ **[2 Marks]**

Solution:

We need to integrate the function $\frac{1}{\sqrt{9-25x^2}}$

Let $5x = t$

$\therefore 5dx = dt$ **[$\frac{1}{2}$ Mark]**

$\Rightarrow \int \frac{1}{\sqrt{9-25x^2}} dx = \frac{1}{5}\int \frac{1}{\sqrt{9-t^2}} dt$

$= \frac{1}{5}\int \frac{1}{\sqrt{3^2-t^2}} dt$

$= \frac{1}{5}\sin^{-1}\left(\frac{t}{3}\right) + C$ [Since, $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a}$]

$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C$ [Substituting t] **[1 Mark]**

Hence, integration of $\frac{1}{\sqrt{9-25x^2}} = \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C$

[$\frac{1}{2}$ Mark]

5. Integrate the function $\frac{3x}{1+2x^4}$ **[2 Marks]**

Solution:

We need to integrate the function $\frac{3x}{1+2x^4}$

Let $\sqrt{2}x^2 = t$

$\therefore 2\sqrt{2}xdx = dt$

[$\frac{1}{2}$ Mark]

$$\Rightarrow \int \frac{3x}{1+2x^4}dx = \frac{3}{2\sqrt{2}}\int \frac{dt}{1+t^2}$$

$$= \frac{3}{2\sqrt{2}}[\tan^{-1}t] + C$$

$= \frac{3}{2\sqrt{2}}\tan^{-1}(\sqrt{2}x^2) + C$ [Substituting t] **[1 Mark]**

Hence, integration of $\frac{3x}{1+2x^4} = \frac{3}{2\sqrt{2}}\tan^{-1}(\sqrt{2}x^2) + C$

[$\frac{1}{2}$ Mark]

6. Integrate the function $\frac{x^2}{1-x^6}$ **[2 Marks]**

Solution:

We need to integrate the function $\frac{x^2}{1-x^6}$

Let $x^3 = t$

$\therefore 3x^2dx = dt$

[$\frac{1}{2}$ Mark]

$$\Rightarrow \int \frac{x^2}{1-x^6}dx = \frac{1}{3}\int \frac{dt}{1-t^2}$$

$$= \frac{1}{3}\left[\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|\right] + C$$

$= \frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right| + C$ [Substituting t] **[1 Mark]**

Hence, integration of $\frac{x^2}{1-x^6} = \frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right| + C$

[$\frac{1}{2}$ Mark]

7. Integrate the function $\frac{x-1}{\sqrt{x^2-1}}$ **[4 Marks]**

Solution:

We need to integrate the function $\frac{x-1}{\sqrt{x^2-1}}$

$\int \frac{x-1}{\sqrt{x^2-1}}dx = \int \frac{x}{\sqrt{x^2-1}}dx - \int \frac{1}{\sqrt{x^2-1}}dx$

[$\frac{1}{2}$ Mark]

For $\int \frac{x}{\sqrt{x^2-1}}dx$, let $x^2 - 1 = t \Rightarrow 2xdx = dt$

$$\therefore \int \frac{x}{\sqrt{x^2-1}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{2}\int t^{-\frac{1}{2}}dt$$

$$= \frac{1}{2}\left[2t^{\frac{1}{2}}\right]$$

$$= \sqrt{t}$$

$= \sqrt{x^2-1}$ [Substituting t] **[1 Mark]**

Hence, we get,

$\int \frac{x-1}{\sqrt{x^2-1}}dx = \int \frac{x}{\sqrt{x^2-1}}dx - \int \frac{1}{\sqrt{x^2-1}}dx$

$= \sqrt{x^2-1} - \log\left|x + \sqrt{x^2-1}\right| + C$ $\left[\int \frac{1}{\sqrt{x^2-a^2}}dt = \log\left|x + \sqrt{x^2-a^2}\right|\right]$ **[2 Marks]**

Hence, integration of $\frac{x-1}{\sqrt{x^2-1}} = \sqrt{x^2-1} - \log\left|x + \sqrt{x^2-1}\right| + C$

[$\frac{1}{2}$ Mark]

8. Integrate the function $\frac{x^2}{\sqrt{x^6+a^6}}$ **[4 Marks]**

Solution:

We need to integrate the function $\frac{x^2}{\sqrt{x^6+a^6}}$

Let $x^3 = t$

$\Rightarrow 3x^2 dx = dt$ **[1 Mark]**

$\therefore \int \frac{x^2}{\sqrt{x^6+a^6}} dx = \frac{1}{3}\int \frac{dt}{\sqrt{t^2+(a^3)^2}}$ **[$\frac{1}{2}$ Mark]**

$= \frac{1}{3}\log\left|t + \sqrt{t^2 + a^6}\right| + C$ $\left[\int \frac{1}{\sqrt{x^2+a^2}} dx = \log\left|x + \sqrt{x^2 + a^2}\right|\right]$

$= \frac{1}{3}\log\left|x^3 + \sqrt{x^6 + a^6}\right| + C$ [Substituting t] **[2 Marks]**

Hence, integration of $\frac{x^2}{\sqrt{x^6+a^6}} = \frac{1}{3}\log\left|x^3 + \sqrt{x^6 + a^6}\right| + C$ **[$\frac{1}{2}$ Mark]**

9. Integrate the function $\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}$ **[4 Marks]**

Solution:

We need to integrate the function $\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}$

Let $\tan x = t$

$\therefore \sec^2 x\, dx = dt$ **[1 Mark]**

$\Rightarrow \int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}} dx = \int \frac{dt}{\sqrt{t^2+2^2}}$ **[$\frac{1}{2}$ Mark]**

$= \log\left|t + \sqrt{t^2 + 4}\right| + \mathrm{C}$ $\left[\int \frac{1}{\sqrt{x^2+a^2}} dx = \log\left|x + \sqrt{x^2 + a^2}\right|\right]$

$= \log\left|\tan x + \sqrt{\tan^2 x + 4}\right| + C$ [Substituting t] **[2 Marks]**

Hence, integration of $\frac{\sec^2 x}{\sqrt{\tan^2 x+4}} = \log\left|\tan x + \sqrt{\tan^2 x + 4}\right| + C$ **[$\frac{1}{2}$ Mark]**

10. Integrate the function $\frac{1}{\sqrt{x^2+2x+2}}$ **[4 Marks]**

Solution:

We need to integrate the function $\frac{1}{\sqrt{x^2+2x+2}}$

$\int \frac{1}{\sqrt{x^2+2x+2}} dx = \int \frac{1}{\sqrt{(x+1)^2+(1)^2}} dx$ **[$\frac{1}{2}$ Mark]**

Let $x + 1 = t$

$\therefore dx = dx$ **[1 Mark]**

$$\Rightarrow \int \frac{1}{\sqrt{x^2 + 2x + 2}} dx = \int \frac{1}{\sqrt{t^2 + 1}} dt$$

$= \log\left|t + \sqrt{t^2 + 1}\right| + C$ $\left[\int \frac{1}{\sqrt{x^2+a^2}} dx = \log\left|x + \sqrt{x^2 + a^2}\right|\right]$

$= \log\left|(x + 1) + \sqrt{(x + 1)^2 + 1}\right| + C$ [Substituting t]

$= \log\left|(x + 1) + \sqrt{x^2 + 2x + 2}\right| + C$ **[2 Marks]**

Hence, integration of $\frac{1}{\sqrt{x^2+2x+2}} = \log\left|(x + 1) + \sqrt{x^2 + 2x + 2}\right| + C$ **[$\frac{1}{2}$ Mark]**

11. Integrate the function $\frac{1}{9x^2+6x+5}$ **[4 Marks]**

Solution:

We need to integrate the function $\frac{1}{9x^2+6x+5}$

$\int \frac{1}{9x^2+6x+5}dx = \int \frac{1}{(3x+1)^2+(2)^2}dx$ **[$\frac{1}{2}$ Mark]**

Let $3x+1=t$

$\therefore 3dx = dt$ **[1 Mark]**

$\Rightarrow \int \frac{1}{(3x+1)^2+(2)^2}dx = \frac{1}{3}\int \frac{1}{t^2+2^2}dt$

$= \frac{1}{3}\times\frac{1}{2}\tan^{-1}\frac{t}{2}+C$ [Since, $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+C$]

$= \frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right)+\mathrm{C}$ [Substituting t] **[2 Marks]**

Hence, integration of $\frac{1}{9x^2+6x+5} = \frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right)+\mathrm{C}$

[$\frac{1}{2}$ Mark]

12. Integrate the function $\frac{1}{\sqrt{7-6x-x^2}}$ **[4 Marks]**

Solution:

We need to integrate the function $\frac{1}{\sqrt{7-6x-x^2}}$

$7-6x-x^2$ can be written as $7-(x^2+6x+9-9)$

Hence,

$7-(x^2+6x+9-9)$

$=16-(x^2+6x+9)$

$=16-(x+3)^2$

$=(4)^2-(x+3)^2$ **[$\frac{1}{2}$ Mark]**

$\therefore \int \frac{1}{\sqrt{7-6x-x^2}}dx = \int \frac{1}{\sqrt{(4)^2-(x+3)^2}}dx$

Let $x+3=t$

$\Rightarrow dx = dt$ **[1 Mark]**

$$\Rightarrow \int \frac{1}{\sqrt{(4)^2-(x+3)^2}}dx = \int \frac{1}{\sqrt{(4)^2-(t)^2}}dt$$

$= \sin^{-1}\left(\frac{t}{4}\right) + C$ [Since, $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} + C$]

$= \sin^{-1}\left(\frac{x+3}{4}\right) + C$ [Substituting t] **[2 Marks]**

Hence, integration of $\frac{1}{\sqrt{7-6x-x^2}} = \sin^{-1}\left(\frac{x+3}{4}\right) + C$

[$\frac{1}{2}$ Mark]

13. Integrate the function $\frac{1}{\sqrt{(x-1)(x-2)}}$ **[4 Marks]**

Solution:

We need to integrate the function $\frac{1}{\sqrt{(x-1)(x-2)}}$

$(x-1)(x-2)$ can be written as $x^2 - 3x + 2$

Hence,

$x^2 - 3x + 2$

$= x^2 - 3x + \frac{9}{4} - \frac{9}{4} + 2$

$= \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}$

$= \left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2$ **[1 Mark]**

$$\therefore \int \frac{1}{\sqrt{(x-1)(x-2)}}dx = \int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}dx$$

Let $x - \frac{3}{2} = t$

$\therefore dx = dt$

$\Rightarrow \int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx = \int \frac{1}{\sqrt{t^2-\left(\frac{1}{2}\right)^2}}dt$ **[$\frac{1}{2}$ Mark]**

$= \log\left[t + \sqrt{t^2 - \left(\frac{1}{2}\right)^2}\right] + C$ $\left[\int \frac{1}{\sqrt{x^2-a^2}}dx = \log\left|x + \sqrt{x^2 - a^2}\right|\right]$

$= \log\left|\left(x - \frac{3}{2}\right) + \sqrt{x^2 - 3x + 2}\right| + C$ [Substituting t] **[2 Marks]**

Hence, integration of $\frac{1}{\sqrt{(x-1)(x-2)}} = \log\left|\left(x - \frac{3}{2}\right) + \sqrt{x^2 - 3x + 2}\right| + C$

[$\frac{1}{2}$ Mark]

14. Integrate the function $\frac{1}{\sqrt{8+3x-x^2}}$ **[4 Marks]**

Solution:

We need to integrate the function $\frac{1}{\sqrt{8+3x-x^2}}$

$8 + 3x - x^2$ can be written as $8 - \left(x^2 - 3x + \frac{9}{4} - \frac{9}{4}\right)$

Hence,

$$8 - \left(x^2 - 3x + \frac{9}{4} - \frac{9}{4}\right)$$

$= \frac{41}{4} - \left(x - \frac{3}{2}\right)^2$ **[1 Mark]**

$$\Rightarrow \int \frac{1}{\sqrt{8 + 3x - x^2}}dx = \int \frac{1}{\sqrt{\frac{41}{4} - \left(x - \frac{3}{2}\right)^2}}dx$$

Let $x - \frac{3}{2} = t$

$\therefore dx = dt$

$\Rightarrow \int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx = \int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-t^2}}dt$ **[1 Mark]**

$= \sin^{-1}\left(\frac{t}{\frac{\sqrt{41}}{2}}\right) + C$ [Since, $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}$]

$= \sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right) + C$ [Substituting t]

Hence, integration of $\frac{1}{\sqrt{8+3x-x^2}} = \sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)$ +C **[2 Marks]**

15. Integrate the function $\frac{1}{\sqrt{(x-a)(x-b)}}$ **[4 Marks]**

Solution:

We need to integrate the function $\frac{1}{\sqrt{(x-a)(x-b)}}$

$(x-a)(x-b)$ can be written as $x^2-(a+b)x+ab$

Therefore,

$x^2-(a+b)x+ab$

$$= x^2-(a+b)x+\frac{(a+b)^2}{4}-\frac{(a+b)^2}{4}+ab$$

$= \left[x-\left(\frac{a+b}{2}\right)\right]^2 - \frac{(a-b)^2}{4}$ **[1 Mark]**

$$\Rightarrow \int \frac{1}{\sqrt{(x-a)(x-b)}}\,dx = \int \frac{1}{\sqrt{\left\{x-\left(\frac{a+b}{2}\right)\right\}^2-\left(\frac{a-b}{2}\right)^2}}\,dx$$

Let $x-\left(\frac{a+b}{2}\right) = t$

$\therefore dx = dt$

$\Rightarrow \int \frac{1}{\sqrt{\left\{x-\left(\frac{a+b}{2}\right)\right\}^2-\left(\frac{a-b}{2}\right)^2}}\,dx = \int \frac{1}{\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}}\,dt$ **[1 Mark]**

$= \log\left|t+\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}\right| + C$ $\left[\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left|x+\sqrt{x^2-a^2}\right|\right]$

$= \log\left|x-\left(\frac{a+b}{2}\right)+\sqrt{(x-a)(x-b)}\right| + C$ [Substituting t]

Hence, integration of $\frac{1}{\sqrt{(x-a)(x-b)}} = \log\left|x - \left(\frac{a+b}{2}\right) + \sqrt{(x-a)(x-b)}\right| + C$ **[2 Marks]**

16. Integrate the function $\frac{4x+1}{\sqrt{2x^2+x-3}}$ **[4 Marks]**

Solution:

We need to integrate the function $\frac{4x+1}{\sqrt{2x^2+x-3}}$

Let $4x + 1 = A\frac{d}{dx}(2x^2 + x - 3) + B$

$\Rightarrow 4x + 1 = A(4x + 1) + B$

$\Rightarrow 4x + 1 = 4Ax + A + B$

Equating the coefficients of x and constant on both sides, we get

$4A = 4 \Rightarrow A = 1$

$A + B = 1 \Rightarrow B = 0$ **[2 Marks]**

Let $2x^2 + x - 3 = t$

$\therefore (4x + 1)dx = dt$

$$\Rightarrow \int \frac{4x+1}{\sqrt{2x^2+x-3}}dx = \int \frac{1}{\sqrt{t}}dt$$

$= 2\sqrt{t} + C$

$= 2\sqrt{2x^2 + x - 3} + C$ [Substituting t]

Hence, integration of $\frac{4x+1}{\sqrt{2x^2+x-3}} = 2\sqrt{2x^2 + x - 3} + C$ **[2 Marks]**

17. Integrate the function $\frac{x+2}{\sqrt{x^2-1}}$ **[4 Marks]**

Solution:

We need to integrate the function $\frac{x+2}{\sqrt{x^2-1}}$

Let $x+2 = A\frac{d}{dx}(x^2-1)+B \ldots(1)$

$\Rightarrow x+2 = A(2x)+B$

Equating the coefficients of x and constant term on both sides, we obtain

$2A = 1 \Rightarrow A = \frac{1}{2}$

$B = 2$

From (1), we obtain

$(x+2) = \frac{1}{2}(2x)+2$ **[1 Mark]**

Then, $\int \frac{x+2}{\sqrt{x^2-1}}dx = \int \frac{\frac{1}{2}(2x)+2}{\sqrt{x^2-1}}dx$

$= \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx + \int \frac{2}{\sqrt{x^2-1}}dx$

For $\frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx$, Let $x^2-1 = t \Rightarrow 2xdx = dt$

Hence, $\frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$

$= \frac{1}{2}[2\sqrt{t}]$

$= \sqrt{t}$

$= \sqrt{x^2-1}$.......(2) [Substituting t] **[1 Mark]**

Again, $\int \frac{2}{\sqrt{x^2-1}}dx = 2\int \frac{1}{\sqrt{x^2-1}}dx = 2\log\left|x+\sqrt{x^2-1}\right|$ **[1 Mark]**

Hence, $\int \frac{x+2}{\sqrt{x^2-1}}dx = \sqrt{x^2-1} + 2\log\left|x+\sqrt{x^2-1}\right| + C$ **[1 Mark]**

18. Integrate the function $\frac{5x-2}{1+2x+3x^2}$ **[6 Marks]**

Solution:

We need to integrate the function $\frac{5x-2}{1+2x+3x^2}$

Let $5x-2=A\frac{d}{dx}(1+2x+3x^2)+B$

$\Rightarrow 5x-2=A(2+6x)+B$

Equating the coefficient of x and constant term on both sides, we obtain

$5=6A\Rightarrow A=\frac{5}{6}$

$2A+B=-2\Rightarrow B=-\frac{11}{3}$

$\therefore 5x-2=\frac{5}{6}(2+6x)+\left(-\frac{11}{3}\right)$ **[1 Mark]**

$$\Rightarrow \int\frac{5x-2}{1+2x+3x^2}dx=\int\frac{\frac{5}{6}(2+6x)-\frac{11}{3}}{1+2x+3x^2}dx$$

$$=\frac{5}{6}\int\frac{2+6x}{1+2x+3x^2}dx-\frac{11}{3}\int\frac{1}{1+2x+3x^2}dx$$

Let $I_1=\int\frac{2+6x}{1+2x+3x^2}dx$ and $I_2=\int\frac{1}{1+2x+3x^2}dx$

$\therefore \int\frac{5x-2}{1+2x+3x^2}dx=\frac{5}{6}I_1-\frac{11}{3}I_2 \ldots(1)$ **[1 Mark]**

$$I_1=\int\frac{2+6x}{1+2x+3x^2}dx$$

Let $1+2x+3x^2=t$ **[$\frac{1}{2}$ Mark]**

$\Rightarrow (2+6x)dx=dt$

$$\therefore I_1=\int\frac{dt}{t}$$

$I_1=\log|t|$

$I_1=\log|1+2x+3x^2| \ldots(2)$ [Substituting t] **[1 Mark]**

For $I_2=\int\frac{1}{1+2x+3x^2}dx$

$$1+2x+3x^2=1+3\left(x^2+\frac{2}{3}x\right)$$

$$=1+3\left(x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}\right)$$

$= 1 + 3\left(x + \frac{1}{3}\right)^2 - \frac{1}{3}$

$= \frac{2}{3} + 3\left(x + \frac{1}{3}\right)^2$

$= 3\left[\left(x + \frac{1}{3}\right)^2 + \frac{2}{9}\right]$

$= 3\left[\left(x + \frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2\right]$

Hence, $I_2 = \frac{1}{3}\int \frac{1}{\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]} dx$

$= \frac{1}{3}\left[\frac{1}{\frac{\sqrt{2}}{3}}\tan^{-1}\left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right)\right]$ [Since, $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$]

$= \frac{1}{3}\left[\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right]$

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)$ ... (3) **[2 Marks]**

Substituting equations (2) and (3) in equation (1), we obtain

$\int \frac{5x-2}{1+2x+3x^2}dx = \frac{5}{6}[\log|1+2x+3x^2|] - \frac{11}{3}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right] + C$

$= \frac{5}{6}\log|1+2x+3x^2| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$

Hence, integration of $\frac{5x-2}{1+2x+3x^2} = \frac{5}{6}\log|1+2x+3x^2| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$

[$\frac{1}{2}$ Mark]

19. Integrate the function $\frac{6x+7}{\sqrt{(x-5)(x-4)}}$ **[6 Marks]**

Solution:

We need to integrate the function $\frac{6x+7}{\sqrt{(x-5)(x-4)}}$

$\frac{6x+7}{\sqrt{(x-5)(x-4)}} = \frac{6x+7}{\sqrt{x^2-9x+20}}$

Let $6x + 7 = A\frac{d}{dx}(x^2 - 9x + 20) + B$

$\Rightarrow 6x + 7 = A(2x - 9) + B$

Equating the coefficients of x and constant term, we obtain

$2A = 6 \Rightarrow A = 3$

$-9A + B = 7 \Rightarrow B = 34$

$\therefore 6x + 7 = 3(2x - 9) + 34$ **[1 Mark]**

$$\int \frac{6x + 7}{\sqrt{x^2 - 9x + 20}} = \int \frac{3(2x - 9) + 34}{\sqrt{x^2 - 9x + 20}} dx$$

$= 3\int \frac{2x-9}{\sqrt{x^2-9x+20}} dx + 34\int \frac{1}{\sqrt{x^2-9x+20}} dx$

Let $I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}}$ and $I_2 = \int \frac{1}{\sqrt{x^2-9x+20}} dx$

$\therefore \int \frac{6x+7}{\sqrt{x^2-9x+20}} = 3I_1 + 34I_2 \dots (1)$ **[1 Mark]**

Now,

$$I_1 = \int \frac{2x - 9}{\sqrt{x^2 - 9x + 20}} dx$$

Let $x^2 - 9x + 20 = t$ **[$\frac{1}{2}$ Mark]**

$\Rightarrow (2x - 9)dx = dt$

$\Rightarrow I_1 = \int \frac{dt}{\sqrt{t}}$

$= 2\sqrt{t}$

Hence, $I_1 = 2\sqrt{x^2 - 9x + 20} \dots (2)$ [Substituting t] **[1 Mark]**

and $I_2 = \int \frac{1}{\sqrt{x^2-9x+20}} dx$

$x^2 - 9x + 20$ can be written as $x^2 - 9x + 20 + \frac{81}{4} - \frac{81}{4}$.

Hence,

$$x^2 - 9x + 20 + \frac{81}{4} - \frac{81}{4}$$

$= \left(x - \frac{9}{2}\right)^2 - \frac{1}{4}$

$= \left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2$

$\Rightarrow I_2 = \int \frac{1}{\sqrt{\left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx$

$= \log\left|\left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20}\right| \ldots (3)$ $\left[\int \frac{1}{\sqrt{x^2 - a^2}} dx = \log\left|x + \sqrt{x^2 - a^2}\right|\right]$ **[2 Marks]**

Substituting equations (2) and (3) in (1), we obtain

$\int \frac{6x+7}{\sqrt{x^2 - 9x + 20}} dx = 3\left[2\sqrt{x^2 - 9x + 20}\right] + 34\log\left[\left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20}\right] + C$

$= 6\sqrt{x^2 - 9x + 20} + 34\log\left|\left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20}\right| + C$

Hence, integration of $\frac{6x+7}{\sqrt{(x-5)(x-4)}} = 6\sqrt{x^2 - 9x + 20} + 34\log\left|\left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20}\right| + C$

[$\frac{1}{2}$ Mark]

20. Integrate the function $\frac{x+2}{\sqrt{4x - x^2}}$ **[6 Marks]**

Solution:

We need to integrate the function $\frac{x+2}{\sqrt{4x - x^2}}$

Let $x + 2 = A\frac{d}{dx}(4x - x^2) + B$

$\Rightarrow x + 2 = A(4 - 2x) + B$

Equating the coefficients of x and constant term on both sides, we obtain

$-2A = 1 \Rightarrow A = -\frac{1}{2}$

$4A + B = 2 \Rightarrow B = 4$

$\Rightarrow (x + 2) = -\frac{1}{2}(4 - 2x) + 4$ **[1 Mark]**

$\therefore \int \frac{x+2}{\sqrt{4x - x^2}} dx = \int \frac{-\frac{1}{2}(4-2x)+4}{\sqrt{4x - x^2}} dx$

$= -\frac{1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}}dx + 4\int \frac{1}{\sqrt{4x-x^2}}dx$

Let $I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}dx$ and $I_2 = \int \frac{1}{\sqrt{4x-x^2}}dx$ ... (1)

$\therefore \int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}I_1 + 4I_2$ **[1 Mark]**

Now, $I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}dx$

Let $4x - x^2 = t$

$\Rightarrow (4-2x)dx = dt$

$\Rightarrow I_1 = \int \frac{1}{\sqrt{t}}dt = 2\sqrt{t} = 2\sqrt{4x-x^2}$ ...(2) [Substituting t] **[1 Mark]**

For $I_2 = \int \frac{1}{\sqrt{4x-x^2}}dx$

$4x - x^2 = -(-4x + x^2)$

$= -(-4x + x^2 + 4 - 4)$

$= 4 - (x-2)^2$

$= (2)^2 - (x-2)^2$ **[$\frac{1}{2}$ Mark]**

Hence, $I_2 = \int \frac{1}{\sqrt{(2)^2-(x-2)^2}}dx$

$= \sin^{-1}\left(\frac{x-2}{2}\right)$ ...(3) [Since, $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} + C$] **[1 Mark]**

Using equations (2) and (3) in (1), we get

$$\int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}\left(2\sqrt{4x-x^2}\right) + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$$

$= -\sqrt{4x-x^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$ **[1 Mark]**

Hence, integration of $\frac{x+2}{\sqrt{4x-x^2}} = -\sqrt{4x-x^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$ **[$\frac{1}{2}$ Mark]**

21. Integrate the function $\frac{x+2}{\sqrt{x^2+2x+3}}$ **[6 Marks]**

Solution:

We need to integrate the function $\frac{x+2}{\sqrt{x^2+2x+3}}$

$\int \frac{(x+2)}{\sqrt{x^2+2x+3}} dx = \frac{1}{2} \int \frac{2(x+2)}{\sqrt{x^2+2x+3}} dx$

$= \frac{1}{2} \int \frac{2x+4}{\sqrt{x^2+2x+3}} dx$

$= \frac{1}{2} \int \frac{2x+2}{\sqrt{x^2+2x+3}} dx + \frac{1}{2} \int \frac{2}{\sqrt{x^2+2x+3}} dx$

$= \frac{1}{2} \int \frac{2x+2}{\sqrt{x^2+2x+3}} dx + \int \frac{1}{\sqrt{x^2+2x+3}} dx$ **[1 Mark]**

Let $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}} dx$ and $I_2 = \int \frac{1}{\sqrt{x^2+2x+3}} dx$

$\therefore \int \frac{x+2}{\sqrt{x^2+2x+3}} dx = \frac{1}{2} I_1 + I_2 \ldots (1)$ **[1 Mark]**

Now, $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}} dx$

Let $x^2 + 2x + 3 = t$

$\Rightarrow (2x + 2)dx = dt$

$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2 + 2x + 3} \ldots (2)$ [Substituting t] **[1 Mark]**

$I_2 = \int \frac{1}{\sqrt{x^2+2x+3}} dx$

$\Rightarrow x^2 + 2x + 3 = x^2 + 2x + 1 + 2 = (x + 1)^2 + (\sqrt{2})^2$ **[$\frac{1}{2}$ Mark]**

$\therefore I_2 = \int \frac{1}{\sqrt{(x+1)^2+(\sqrt{2})^2}} dx = \log\left|(x + 1) + \sqrt{x^2 + 2x + 3}\right| \ldots (3)$ **[1 Mark]**

Using equations (2) and (3) in (1), we obtain

$\int \frac{x+2}{\sqrt{x^2+2x+3}} dx = \frac{1}{2}\left[2\sqrt{x^2 + 2x + 3}\right] + \log\left|(x + 1) + \sqrt{x^2 + 2x + 3}\right| + C$

$= \sqrt{x^2 + 2x + 3} + \log\left|(x + 1) + \sqrt{x^2 + 2x + 3}\right| + C$ **[1 Mark]**

Hence, integration of $\frac{x+2}{\sqrt{x^2+2x+3}} = \sqrt{x^2+2x+3} + \log\left|(x+1) + \sqrt{x^2+2x+3}\right| + C$

[$\frac{1}{2}$ Mark]

22. Integrate the function $\frac{x+3}{x^2-2x-5}$ **[6 Marks]**

Solution:

We need to integrate the function $\frac{x+3}{x^2-2x-5}$

Let $(x+3) = A\frac{d}{dx}(x^2-2x-5) + B$

$(x+3) = A(2x-2) + B$

Equating the coefficients of x and constant term on both sides, we obtain

$2A = 1 \Rightarrow A = \dfrac{1}{2}$

$-2A + B = 3 \Rightarrow B = 4$

$\therefore (x+3) = \frac{1}{2}(2x-2) + 4$ **[1 Mark]**

$$\Rightarrow \int \frac{x+3}{x^2-2x-5}dx = \int \frac{\frac{1}{2}(2x-2)+4}{x^2-2x-5}dx$$

$$= \frac{1}{2}\int \frac{2x-2}{x^2-2x-5}dx + 4\int \frac{1}{x^2-2x-5}dx$$

Let $I_1 = \int \frac{2x-2}{x^2-2x-5}dx$ and $I_2 = \int \frac{1}{x^2-2x-5}dx$

$\therefore \int \frac{x+3}{(x^2-2x-5)}dx = \frac{1}{2}I_1 + 4I_2 \ldots (1)$ **[1 Mark]**

Now, $I_1 = \int \frac{2x-2}{x^2-2x-5}dx$

Let $x^2 - 2x - 5 = t$

$\Rightarrow (2x-2)dx = dt$ **[$\frac{1}{2}$ Mark]**

$\Rightarrow I_1 = \int \frac{dt}{t} = \log|t| = \log|x^2-2x-5| \ldots (2)$ [Substituting t] **[1 Mark]**

$$I_2 = \int \frac{1}{x^2 - 2x - 5} dx$$

$$= \int \frac{1}{(x^2 - 2x + 1) - 6} dx$$

$$= \int \frac{1}{(x-1)^2 - (\sqrt{6})^2} dx$$

$= \frac{1}{2\sqrt{6}} \log\left(\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right) \ldots (3)$ **[1 Mark]**

Substituting (2) and (3) in (1), we obtain

$$\int \frac{x+3}{x^2 - 2x - 5} dx = \frac{1}{2}\log|x^2 - 2x - 5| + \frac{4}{2\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$$

$= \frac{1}{2}\log|x^2 - 2x - 5| + \frac{2}{\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$ **[1 Mark]**

Hence, integration of $\frac{x+3}{x^2-2x-5} = \frac{1}{2}\log|x^2 - 2x - 5| + \frac{2}{\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$

[$\frac{1}{2}$ Mark]

23. Integrate the function $\frac{5x+3}{\sqrt{x^2+4x+10}}$ **[6 Marks]**

Solution:

We need to integrate the function $\frac{5x+3}{\sqrt{x^2+4x+10}}$

Let $5x + 3 = A\frac{d}{dx}(x^2 + 4x + 10) + B$

$\Rightarrow 5x + 3 = A(2x + 4) + B$

Equating the coefficients of x and constant term, we obtain

$$2A = 5 \Rightarrow A = \frac{5}{2}$$

$$4A + B = 3 \Rightarrow B = -7$$

$\therefore 5x + 3 = \frac{5}{2}(2x + 4) - 7$ **[1 Mark]**

$$\Rightarrow \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \int \frac{\frac{5}{2}(2x+4)-7}{\sqrt{x^2+4x+10}}dx$$

$$= \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2+4x+10}}dx - 7\int \frac{1}{\sqrt{x^2+4x+10}}dx$$

Let $I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}}dx$ and $I_2 = \int \frac{1}{\sqrt{x^2+4x+10}}dx$

$\therefore \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}I_1 - 7I_2 \ldots (1)$ **[1 Mark]**

Now, $I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}}dx$

Let $x^2+4x+10 = t$

$\therefore (2x+4)dx = dt$ **[$\frac{1}{2}$ Mark]**

$\Rightarrow I_1 = \int \frac{dt}{t} = 2\sqrt{t} = 2\sqrt{x^2+4x+10} \ldots (2)$ [Substituting t] **[1 Mark]**

$$I_2 = \int \frac{1}{\sqrt{x^2+4x+10}}dx$$

$$= \int \frac{1}{\sqrt{(x^2+4x+4)+6}}dx$$

$$= \int \frac{1}{(x+2)^2+\left(\sqrt{6}\right)^2}dx$$

$= \log\left|(x+2)\sqrt{x^2+4x+10}\right| \ldots (3)$ **[1 Mark]**

Using equations (2) and (3) in (1), we obtain

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}\left[2\sqrt{x^2+4x+10}\right] - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$$

$= 5\sqrt{x^2+4x+10} - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$ **[1 Mark]**

Hence, integration of $\frac{5x+3}{\sqrt{x^2+4x+10}} = 5\sqrt{x^2+4x+10} - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$

[$\frac{1}{2}$ Mark]

24. $\int \frac{dx}{x^2+2x+2}$ equals **[2 Marks]**

(A) $x\tan^{-1}(x+1)+C$

(B) $\tan^{-1}(x+1)+C$

(C) $(x+1)\tan^{-1}x+C$

(D) $\tan^{-1}x+C$

Solution:

Given: $\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x^2+2x+1)+1}$

$= \int \frac{1}{(x+1)^2+(1)^2}dx$

[$\frac{1}{2}$ Mark]

Hence, $\int \frac{dx}{x^2+2x+2} = [\tan^{-1}(x+1)] + C$ **[1 Mark]**

Hence, the correct answer is B

[$\frac{1}{2}$ Mark]

25. $\int \frac{dx}{\sqrt{9x-4x^2}}$ equals **[4 Marks]**

(A) $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

(B) $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$

(C) $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

(D) $\frac{1}{2}\sin^{-1}\left(\frac{9x-8}{9}\right)+C$

Solution:

Given: $\int \frac{dx}{\sqrt{9x-4x^2}}$

$= \int \frac{1}{\sqrt{-4\left(x^2-\frac{9}{4}x\right)}} dx$

$= \int \frac{1}{-4\left(x^2-\frac{9}{4}x+\frac{81}{64}-\frac{81}{64}\right)} dx$

$= \int \frac{1}{\sqrt{-4\left[\left(x-\frac{9}{8}\right)^2-\left(\frac{9}{8}\right)^2\right]}} dx$

$= \frac{1}{2} \int \frac{1}{\sqrt{\left(\frac{9}{8}\right)^2-\left(x-\frac{9}{8}\right)^2}} dx$ **[1 Mark]**

$= \frac{1}{2}\left[\sin^{-1}\left(\frac{x-\frac{9}{8}}{\frac{9}{8}}\right)\right] + C$ $\left(\int \frac{dy}{\sqrt{a^2-y^2}} = \sin^{-1}\frac{y}{a} + C\right)$

$= \frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right) + C$ **[2 Marks]**

Hence, $\int \frac{dx}{\sqrt{9x-4x^2}} = \frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right) + C$

Hence, the correct answer is B. **[1 Mark]**

Exercise 7.5

1. Integrate the rational function $\frac{x}{(x+1)(x+2)}$ **[4 Marks]**

Solution:

We need to integrate $\frac{x}{(x+1)(x+2)}$

Let $\frac{x}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{x+2}$ **[$\frac{1}{2}$ Mark]**

$\Rightarrow x = A(x+2) + B(x+1)$

Equating the coefficients of x and constant term, we get

$A + B = 1$

$2A + B = 0$

On solving we get

$A = -1$ and $B = 2$ **[1 Mark]**

$\therefore \frac{x}{(x+1)(x+2)} = \frac{-1}{(x+1)} + \frac{2}{(x+2)}$ **[$\frac{1}{2}$ Mark]**

$$\Rightarrow \int \frac{x}{(x+1)(x+2)} dx = \int \frac{-1}{(x+1)} + \frac{2}{(x+2)} dx$$

$= -\log|x+1| + 2\log|x+2| + C$

$= \log(x+2)^2 - \log|x+1| + C$

$= \log\frac{(x+2)^2}{|x+1|} + C$

Hence, Integration of $\frac{x}{(x+1)(x+2)} = \log\frac{(x+2)^2}{|x+1|} + C$ **[2 Marks]**

2. Integrate the rational function $\frac{1}{x^2-9}$ **[4 Marks]**

Solution:

We need to integrate $\frac{1}{x^2-9}$

Let $\frac{1}{(x+3)(x-3)} = \frac{A}{(x+3)} + \frac{B}{(x-3)}$ **[$\frac{1}{2}$ Mark]**

$\Rightarrow 1 = A(x-3) + B(x+3)$

Equating the coefficients of x and constant term, we get

$A + B = 0$

$-3A + 3B = 1$

On solving, we get

$A = -\frac{1}{6}$ and $B = \frac{1}{6}$ **[1 Mark]**

$\therefore \frac{1}{(x+3)(x-3)} = \frac{-1}{6(x+3)} + \frac{1}{6(x-3)}$ [$\frac{1}{2}$ Mark]

$$\Rightarrow \int \frac{1}{(x^2-9)} dx = \int \left(\frac{-1}{6(x+3)} + \frac{1}{6(x-3)}\right) dx$$

$$= -\frac{1}{6}\log|x+3| + \frac{1}{6}\log|x-3| + C$$

$$= \frac{1}{6}\log\left|\frac{(x-3)}{(x+3)}\right| + C$$

Hence, Integration of $\frac{1}{x^2-9} = \frac{1}{6}\log\left|\frac{(x-3)}{(x+3)}\right| + C$ [2 Marks]

3. Integrate the rational function $\frac{3x-1}{(x-1)(x-2)(x-3)}$ [4 Marks]

Solution:

We need to integrate $\frac{3x-1}{(x-1)(x-2)(x-3)}$

Let $\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$ [$\frac{1}{2}$ Mark]

$$3x-1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \ldots (1)$$

Equating the coefficients of x^2, x and constant term, we get

$$A + B + C = 0$$

$$-5A - 4B - 3C = 3$$

$$6A + 3B + 2C = -1$$

Solving these equations, we get

$A = 1, B = -5,$ and $C = 4$ [1 Mark]

$\therefore \frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{1}{(x-1)} - \frac{5}{(x-2)} + \frac{4}{(x-3)}$ [$\frac{1}{2}$ Mark]

$$\Rightarrow \int \frac{3x-1}{(x-1)(x-2)(x-3)} dx = \int \left\{\frac{1}{(x-1)} - \frac{5}{(x-2)} + \frac{4}{(x-3)}\right\} dx$$

$= \log|x-1| - 5\log|x-2| + 4\log|x-3| + C$

Hence, Integration of $\frac{3x-1}{(x-1)(x-2)(x-3)} = \log|x-1| - 5\log|x-2| + 4\log|x-3| + C$ **[2 Marks]**

4. Integrate the rational function $\frac{x}{(x-1)(x-2)(x-3)}$ **[4 Marks]**

Solution:

We need to integrate $\frac{x}{(x-1)(x-2)(x-3)}$

Let $\frac{x}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$ **[$\frac{1}{2}$ Mark]**

$\Rightarrow x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$ … (1)

Equating the coefficients of x^2, x and constant term, we get

$A + B + C = 0$

$-5A - 4B - 3C = 1$

$6A + 4B + 2C = 0$

Solving these equations, we get

$A = \frac{1}{2}, B = -2$, and $C = \frac{3}{2}$ **[1 Mark]**

$\therefore \frac{x}{(x-1)(x-2)(x-3)} = \frac{1}{2(x-1)} - \frac{2}{(x-2)} + \frac{3}{2(x-3)}$ **[$\frac{1}{2}$ Mark]**

$$\Rightarrow \int \frac{x}{(x-1)(x-2)(x-3)} dx = \int \left\{\frac{1}{2(x-1)} - \frac{2}{(x-2)} + \frac{3}{2(x-3)}\right\} dx$$

$= \frac{1}{2}\log|x-1| - 2\log|x-2| + \frac{3}{2}\log|x-3| + C$

Hence, integration of $\frac{x}{(x-1)(x-2)(x-3)} = \frac{1}{2}\log|x-1| - 2\log|x-2| + \frac{3}{2}\log|x-3| + C$ **[2 Marks]**

5. Integrate the rational function $\frac{2x}{x^2+3x+2}$ **[4 Marks]**

Solution:

We need to integrate: $\frac{2x}{x^2+3x+2}$

Let $\frac{2x}{x^2+3x+2} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$ **[$\frac{1}{2}$ Mark]**

$\Rightarrow 2x = A(x+2) + B(x+1)$ ... (1)

Equating the coefficients of x^2, x and constant term, we get

$A + B = 2$

$2A + B = 0$

Solving these equations, we get

$A = -2$ and $B = 4$ **[1 Mark]**

$\therefore \frac{2x}{(x+1)(x+2)} = \frac{-2}{(x+1)} + \frac{4}{(x+2)}$ **[$\frac{1}{2}$ Mark]**

$\Rightarrow \int \frac{2x}{(x+1)(x+2)} dx = \int \left\{\frac{4}{(x+2)} - \frac{2}{(x+1)}\right\} dx$

$= 4\log|x+2| - 2\log|x+1| + C$

Hence, Integration of $\frac{2x}{x^2+3x+2} = 4\log|x+2| - 2\log|x+1| + C$ **[2 Marks]**

6. Integrate the rational function $\frac{1-x^2}{x(1-2x)}$ **[4 Marks]**

Solution:

We need to integrate $\frac{1-x^2}{x(1-2x)}$

It is seen that the given integrand is not a proper fraction. Therefore, on dividing $(1-x^2)$ by $x(1-2x)$, we get

$\frac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{1}{2}\left(\frac{2-x}{x(1-2x)}\right)$ ...(1)

Let $\frac{2-x}{x(1-2x)} = \frac{A}{x} + \frac{B}{(1-2x)}$ [$\frac{1}{2}$ Mark]

$\Rightarrow (2-x) = A(1-2x) + Bx$

Equating the coefficients of x^2, x and constant term, we get

$-2A + B = -1$

And $A = 2$

Solving these equations, we get

$A = 2$ and $B = 3$ [1 Mark]

$\therefore \frac{2-x}{x(1-2x)} = \frac{2}{x} + \frac{3}{1-2x}$ [$\frac{1}{2}$ Mark]

Substituting in equation (1), we get

$$\frac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{1}{2}\left\{\frac{2}{x} + \frac{2}{(1-2x)}\right\}$$

$$\Rightarrow \int \frac{1-x^2}{x(1-2x)}dx = \int \left\{\frac{1}{2} + \frac{1}{2}\left(\frac{2}{x} + \frac{3}{1-2x}\right)\right\}dx$$

$$= \frac{x}{2} + \log|x| + \frac{3}{2(-2)}\log|1-2x| + C$$

$$= \frac{x}{2} + \log|x| - \frac{3}{4}\log|1-2x| + C$$

Hence, Integration of $\frac{1-x^2}{x(1-2x)} = \frac{x}{2} + \log|x| - \frac{3}{4}\log|1-2x| + C$ [2 Marks]

7. Integrate the rational function $\frac{x}{(x^2+1)(x-1)}$ [6 Marks]

Solution:

We need to integrate $\frac{x}{(x^2+1)(x-1)}$

Let $\frac{x}{(x^2+1)(x-1)} = \frac{Ax+B}{(x^2+1)} + \frac{C}{(x-1)}$ ...(1) [$\frac{1}{2}$ Mark]

$\Rightarrow x = (Ax+B)(x-1) + C(x^2+1)$

$\Rightarrow x = Ax^2 - Ax + Bx - B + Cx^2 + C$

Equating the coefficient of $x^2, x,$ and constant term, we get

$A + C = 0$

$-A + B = 1$

$-B + C = 0$

On solving these equations, we get

$A = -\frac{1}{2}, B = \frac{1}{2}, \text{and } C = \frac{1}{2}$ [2 Marks]

From equation (1), we get

$\therefore \frac{x}{(x^2+1)(x-1)} = \frac{\left(-\frac{1}{2}x+\frac{1}{2}\right)}{x^2+1} + \frac{\frac{1}{2}}{(x-1)}$ [$\frac{1}{2}$ Mark]

$$\Rightarrow \int \frac{x}{(x^2+1)(x-1)}dx = -\frac{1}{2}\int \frac{x}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x-1}dx$$

$$= -\frac{1}{4}\int \frac{2x}{x^2+1}dx + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x-1| + C$$

Consider $\int \frac{2x}{x^2+1}dx$, let $(x^2+1) = t \Rightarrow 2xdx = dt$

$$\Rightarrow \int \frac{2x}{x^2+1}dx = \int \frac{dt}{t} = \log|t| = \log|x^2+1|$$

$$\therefore \int \frac{x}{(x^2+1)(x-1)} = -\frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x-1| + C$$

$= \frac{1}{2}\log|x-1| - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + C$

Hence, Integration of $\frac{x}{(x^2+1)(x-1)} = \frac{1}{2}\log|x-1| - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + C$ [3 Marks]

8. Integrate the rational function $\frac{x}{(x-1)^2(x+2)}$ [6 Marks]

Solution:

We need to integrate $\frac{x}{(x-1)^2(x+2)}$

Let $\frac{x}{(x-1)^2(x+2)}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x+2)}$ **[$\frac{1}{2}$ Mark]**

$\Rightarrow x=A(x-1)(x+2)+B(x+2)+C(x-1)^2$

Equating the coefficients of x^2, x and constant term, we get

$A+C=0$

$A+B-2C=1$

$-2A+2B+C=0$

On solving, we get

$A=\frac{2}{9}$ and $C=\frac{-2}{9}$

$B=\frac{1}{3}$ **[2 Marks]**

$\therefore \frac{x}{(x-1)^2(x+2)}=\frac{2}{9(x-1)}+\frac{1}{3(x-1)^2}-\frac{2}{9(x+2)}$ **[$\frac{1}{2}$ Mark]**

$$\Rightarrow \int \frac{x}{(x-1)^2(x+2)}dx=\frac{2}{9}\int \frac{1}{(x-1)}dx+\frac{1}{3}\int \frac{1}{(x-1)^2}dx-\frac{2}{9}\int \frac{1}{(x+2)}dx$$

$$=\frac{2}{9}\log|x-1|+\frac{1}{3}\left(\frac{-1}{x-1}\right)-\frac{2}{9}\log|x+2|+C$$

$$=\frac{2}{9}\log\left|\frac{x-1}{x+2}\right|-\frac{1}{3(x-1)}+C$$

Hence, Integration of $\frac{x}{(x-1)^2(x+2)}=\frac{2}{9}\log\left|\frac{x-1}{x+2}\right|-\frac{1}{3(x-1)}+C$ **[3 Marks]**

9. Integrate the rational function $\frac{3x+5}{x^3-x^2-x+1}$ **[6 Marks]**

Solution:

We need to integrate $\frac{3x+5}{x^3-x^2-x+1}$

$\frac{3x+5}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$ **[$\frac{1}{2}$ Mark]**

$\Rightarrow 3x + 5 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$

$\Rightarrow 3x + 5 = A(x^2 - 1) + B(x+1) + C(x^2 + 1 - 2x) \ldots (1)$

Equating the coefficients of x^2, x and constant term, we get

$A + C = 0$

$B - 2C = 3$

$-A + B + C = 5$

On solving, we get

$B = 4$

$A = -\frac{1}{2}$ and $C = \frac{1}{2}$ **[2 Marks]**

$\therefore \frac{3x+5}{(x-1)^2(x+1)} = \frac{-1}{2(x-1)} + \frac{4}{(x-1)^2} + \frac{1}{2(x+1)}$ **[$\frac{1}{2}$ Mark]**

$$\Rightarrow \int \frac{3x+5}{(x-1)^2(x+1)} dx = -\frac{1}{2}\int \frac{1}{x-1} dx + 4\int \frac{1}{(x-1)^2} dx + \frac{1}{2}\int \frac{1}{(x+1)} dx$$

$$= -\frac{1}{2}\log|x-1| + 4\left(\frac{-1}{x-1}\right) + \frac{1}{2}\log|x+1| + C$$

$= \frac{1}{2}\log\left|\frac{x+1}{x-1}\right| - \frac{4}{x-1} + C$

Hence, Integration of $\frac{3x+5}{x^3-x^2-x+1} = \frac{1}{2}\log\left|\frac{x+1}{x-1}\right| - \frac{4}{x-1} + C$ **[3 Marks]**

10. Integrate the rational function $\frac{2x-3}{(x^2-1)(2x+3)}$ **[6 Marks]**

Solution:

Given: $\frac{2x-3}{(x^2-1)(2x+3)} = \frac{2x-3}{(x+1)(x-1)(2x+3)}$

Let $\frac{2x-3}{(x+1)(x-1)(2x+3)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(2x+3)}$ [$\frac{1}{2}$ Mark]

$\Rightarrow (2x-3) = A(x-1)(2x+3) + B(x+1)(2x+3) + C(x+1)(x-1)$

$\Rightarrow (2x-3) = A(2x^2+x-3) + B(2x^2+5x+3) + C(x^2-1)$

$\Rightarrow (2x-3) = (2A+2B+C)x^2 + (A+5B)x + (-3A+3B-C)$

Equating the coefficients of x^2, x and constant, we get

$2A + 2B + C = 0$

$A + 5B = 2$

$-3A + 3B - C = -3$

On solving, we get

$B = -\frac{1}{10}, A = \frac{5}{2},$ and $C = -\frac{24}{5}$ [2 Marks]

$\therefore \frac{2x-3}{(x+1)(x-1)(2x+3)} = \frac{5}{2(x+1)} - \frac{1}{10(x-1)} - \frac{24}{5(2x+3)}$ [$\frac{1}{2}$ Mark]

$$\Rightarrow \int \frac{2x-3}{(x^2-1)(2x+3)} dx = \frac{5}{2}\int \frac{1}{(x+1)} dx - \frac{1}{10}\int \frac{1}{x-1} dx - \frac{24}{5}\int \frac{1}{(2x+3)} dx$$

$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{24}{5 \times 2}\log|2x+3|$$

$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{12}{5}\log|2x+3| + C$

Hence, Integration of $\frac{2x-3}{(x^2-1)(2x+3)} = \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{12}{5}\log|2x+3| + C$ [3 Marks]

11. Integrate the rational function $\frac{5x}{(x+1)(x^2-4)}$ [6 Marks]

Solution:

We need to integrate $\frac{5x}{(x+1)(x^2-4)} = \frac{5x}{(x+1)(x+2)(x-2)}$

Let $\frac{5x}{(x+1)(x+2)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x-2)}$ **[$\frac{1}{2}$ Mark]**

$\Rightarrow 5x = A(x+2)(x-2) + B(x+1)(x-2) + C(x+1)(x+2)$ ... (1)

Equating the coefficients of x^2, x and constant, we get

$A + B + C = 0$

$-B + 3C = 5$ and

$-4A - 2B + 2C = 0$

On solving, we get

$A = \frac{5}{3}, B = -\frac{5}{2}$, and $C = \frac{5}{6}$ **[2 Marks]**

$\therefore \frac{5x}{(x+1)(x+2)(x-2)} = \frac{5}{3(x+1)} - \frac{5}{2(x+2)} + \frac{5}{6(x-2)}$ **[$\frac{1}{2}$ Mark]**

$$\Rightarrow \int \frac{5x}{(x+1)(x^2-4)} dx$$

$$= \frac{5}{3}\int \frac{1}{(x+1)} dx - \frac{5}{2}\int \frac{1}{(x+2)} dx + \frac{5}{6}\int \frac{1}{(x-2)} dx$$

$= \frac{5}{3}\log|x+1| - \frac{5}{2}\log|x+2| + \frac{5}{6}\log|x-2| + C$

Hence, Integration of $\frac{5x}{(x+1)(x^2-4)} = \frac{5}{3}\log|x+1| - \frac{5}{2}\log|x+2| + \frac{5}{6}\log|x-2| + C$ **[3 Marks]**

12. Integrate the rational function $\frac{x^3+x+1}{x^2-1}$ **[4 Marks]**

Solution:

We need to integrate $\frac{x^3+x+1}{x^2-1}$

It is seen that the given integrand is not a proper fraction.

Hence, on dividing $(x^3 + x + 1)$ by $x^2 - 1$, we get

$$\frac{x^3 + x + 1}{x^2 - 1} = x + \frac{2x + 1}{x^2 - 1}$$

Let $\frac{2x+1}{x^2-1} = \frac{A}{(x+1)} + \frac{B}{(x-1)}$ **[$\frac{1}{2}$ Mark]**

$\Rightarrow 2x + 1 = A(x-1) + B(x+1) \ldots (1)$

Equating the coefficients of x and constant, we get

$A + B = 2$

$-A + B = 1$

On solving, we get

$A = \frac{1}{2}$ and $B = \frac{3}{2}$ **[1 Mark]**

$\therefore \frac{x^3+x+1}{x^2-1} = x + \frac{1}{2(x+1)} + \frac{3}{2(x-1)}$ **[$\frac{1}{2}$ Mark]**

$$\Rightarrow \int \frac{x^3+x+1}{x^2-1} dx = \int x\, dx + \frac{1}{2}\int \frac{1}{(x+1)} dx + \frac{3}{2}\int \frac{1}{(x-1)} dx$$

$= \frac{x^2}{2} + \frac{1}{2}\log|x+1| + \frac{3}{2}\log|x-1| + C$

Hence, Integration of $\frac{x^3+x+1}{x^2-1} = \frac{x^2}{2} + \frac{1}{2}\log|x+1| + \frac{3}{2}\log|x-1| + C$ **[2 Marks]**

13. Integrate the rational function $\frac{2}{(1-x)(1+x^2)}$ **[6 Marks]**

Solution:

We need to integrate $\frac{2}{(1-x)(1+x^2)}$

Let $\frac{2}{(1-x)(1+x^2)} = \frac{A}{(1-x)} + \frac{Bx+C}{(1+x^2)}$ **[$\frac{1}{2}$ Mark]**

$\Rightarrow 2 = A(1+x^2) + (Bx+C)(1-x)$

$\Rightarrow 2 = A + Ax^2 + Bx - Bx^2 + C - Cx$

Equating the coefficient of x^2, x, and constant term, we get

$A - B = 0$

$B - C = 0$

$A + C = 0$

On solving these equations, we get

$A = 1, B = 1,$ and $C = 1$ **[2 Marks]**

$\therefore \frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2}$ **[$\frac{1}{2}$ Mark]**

$$\Rightarrow \int \frac{2}{(1-x)(1+x^2)} dx = \int \frac{1}{1-x} dx + \int \frac{x}{1+x^2} dx + \int \frac{1}{1+x^2} dx$$

$$= -\int \frac{1}{x-1} dx + \frac{1}{2}\int \frac{2x}{1+x^2} dx + \int \frac{1}{1+x^2} dx$$

$$= -\log|x-1| + \frac{1}{2}\log|1+x^2| + \tan^{-1}x + C$$

Hence, Integration of $\frac{2}{(1-x)(1+x^2)} = -\log|x-1| + \frac{1}{2}\log|1+x^2| + \tan^{-1}x + C$ **[3 Marks]**

14. Integrate the rational function $\frac{3x-1}{(x+2)^2}$ **[4 Marks]**

Solution:

We need to integrate $\frac{3x-1}{(x+2)^2}$

Let $\frac{3x-1}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$ **[$\frac{1}{2}$ Mark]**

$\Rightarrow 3x - 1 = A(x+2) + B$

Equating the coefficient of x and constant term, we get

$A = 3$

$2A + B = -1 \Rightarrow B = -7$ **[1 Mark]**

$\therefore \frac{3x-1}{(x+2)^2} = \frac{3}{(x+2)} - \frac{7}{(x+2)^2}$ **[$\frac{1}{2}$ Mark]**

$\Rightarrow \int \frac{3x-1}{(x+2)^2}dx = 3\int \frac{1}{(x+2)}dx - 7\int \frac{1}{(x+2)^2}dx$

$= 3\log|x+2| - 7\left(\frac{-1}{(x+2)}\right) + C$

Hence, Integration of $\frac{3x-1}{(x+2)^2} = 3\log|x+2| + \frac{7}{(x+2)} + C$ **[2 Marks]**

15. Integrate the rational function $\frac{1}{(x^4-1)}$ **[6 Marks]**

Solution:

We need to integrate $\frac{1}{(x^4-1)}$

$$\frac{1}{(x^4-1)} = \frac{1}{(x^2-1)(x^2+1)} = \frac{1}{(x+1)(x-1)(1+x^2)}$$

Let $\frac{1}{(x+1)(x-1)(1+x^2)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{Cx+D}{(x^2+1)}$ **[$\frac{1}{2}$ Mark]**

$\Rightarrow 1 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x^2-1)$

$\Rightarrow 1 = A(x^3+x-x^2-1) + B(x^3+x+x^2+1) + Cx^3 + Dx^2 - Cx - D$

$\Rightarrow 1 = (A+B+C)x^3 + (-A+B+D)x^2 + (A+B-C)x + (-A+B-D)$

Equating the coefficient of x^3, x^2, x, and constant term, we get

$A+B+C=0$

$-A+B+D=0$

$A+B-C=0$

$-A+B-D=0$

On solving these equations, we get

$A = -\frac{1}{4}, B = \frac{1}{4}, C = 0$, and $D = -\frac{1}{2}$ **[2 Marks]**

$\therefore \frac{1}{x^4-1} = \frac{-1}{4(x+1)} + \frac{1}{4(x-1)} - \frac{1}{2(x^2+1)}$ **[$\frac{1}{2}$ Mark]**

$$\Rightarrow \int \frac{1}{x^4-1}dx = -\frac{1}{4}\log|x-1| + \frac{1}{4}\log|x-1| - \frac{1}{2}\tan^{-1}x + C$$

$$= \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{-1}x + C$$

Hence, Integration of $\frac{1}{(x^4-1)} = \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{-1}x + C$ **[3 Marks]**

16. Integrate the rational function $\frac{1}{x(x^n+1)}$ **[6 Marks]**

[Hint: multiply numerator and denominator by x^{n-1} and put $x^n = t$]

Solution:

We need to integrate $\frac{1}{x(x^n+1)}$

Multiplying numerator and denominator by x^{n-1}, we get

$$\frac{1}{x(x^n+1)} = \frac{x^{n-1}}{x^{n-1}x(x^n+1)} = \frac{x^{n-1}}{x^n(x^n+1)}$$

Let $x^n = t \Rightarrow nx^{n-1}dx = dt$ **[1 Mark]**

$\therefore \int \frac{1}{x(x^n+1)}dx = \int \frac{x^{n-1}}{x^n(x^n+1)}dx = \frac{1}{n}\int \frac{1}{t(t+1)}dt$ **[$\frac{1}{2}$ Mark]**

Let $\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{(t+1)}$

$\Rightarrow 1 = A(1+t) + Bt \ldots (1)$

Equating the coefficients of t and constant, we get

$A = 1$ and $B = -1$ **[1 Mark]**

$\therefore \frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{(1+t)}$ **[$\frac{1}{2}$ Mark]**

$$\Rightarrow \int \frac{1}{x(x^n+1)}dx = \frac{1}{n}\int \left\{\frac{1}{t} - \frac{1}{(t+1)}\right\}dt$$

$$= \frac{1}{n}[\log|t| - \log|t+1| + C$$

$= \frac{1}{n}[\log|x^n| - \log|x^n+1|] + C$

$= \frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right| + C$

Hence, integration of $\frac{1}{x(x^n+1)} = \frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right| + C$ **[3 Marks]**

17. Integrate the rational function $\frac{\cos x}{(1-\sin x)(2-\sin x))}$ [Hint: Put $\sin x = t$] **[6 Marks]**

Solution:

We need to integrate $\frac{\cos x}{(1-\sin x(2-\sin x))}$

Let $\sin x = t \Rightarrow \cos x\, dx = dt$ **[1 Mark]**

$\therefore \int \frac{\cos x}{(1-\sin x)(2-\sin x)} dx = \int \frac{dt}{(1-t)(2-t)}$

Let $\frac{1}{(1-t)(2-t)} = \frac{A}{1-t} + \frac{B}{(2-t)}$ **[$\frac{1}{2}$ Mark]**

$\Rightarrow 1 = A(2-t) + B(1-t)$ ... (1)

Equating the coefficient of t and constant, we get

$-2A - B = 0$ and

$2A + B = 1$

On solving, we get

$A = 1$ and $B = -1$ **[2 Marks]**

$\therefore \frac{1}{(1-t)(2-t)} = \frac{1}{(1-t)} - \frac{1}{(2-t)}$ **[$\frac{1}{2}$ Mark]**

$\Rightarrow \int \frac{\cos x}{(1-\sin x)(2-\sin x)} dx = \int \left\{\frac{1}{1-t} - \frac{1}{(2-t)}\right\} dt$

$= -\log|1-t| + \log|2-t| + C$

$= \log\left|\frac{2-t}{1-t}\right| + C$

$= \log\left|\frac{2-\sin x}{1-\sin x}\right| + C$

Hence, integration of $\frac{\cos x}{(1-\sin x)(2-\sin x)} = \log\left|\frac{2-\sin x}{1-\sin x}\right| + C$ **[2 Marks]**

18. Integrate the rational function $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$ **[6 Marks]**

Solution:

We need to integrate $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 - \frac{(4x^2+10)}{(x^2+3)(x^2+4)}$$

$\frac{4x^2+10}{(x^2+3)(x^2+4)} = \frac{Ax+B}{(x^2+3)} + \frac{Cx+D}{(x^2+4)}$ **[$\frac{1}{2}$ Mark]**

$\Rightarrow 4x^2 + 10 = (Ax+B)(x^2+4) + (Cx+D)(x^2+3)$

$\Rightarrow 4x^2 + 10 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + 3Cx + Dx^2 + 3D$

$\Rightarrow 4x^2 + 10 = (A+C)x^3 + (B+D)x^2 + (4A+3C)x + (4B+3D)$ **[1 Mark]**

Equating the coefficients of x^3, x^2, x, and constant term, we get

$A + C = 0$

$B + D = 4$

$4A + 3C = 0$

$4B + 3D = 10$

On solving these equations, we get

$A = 0, B = -2, C = 0,$ and $D = 6$ **[2 Marks]**

$\therefore \frac{4x^2+10}{(x^2+3)(x^2+4)} = \frac{-2}{(x^2+3)} + \frac{6}{(x^2+4)}$ [$\frac{1}{2}$ Mark]

Hence, $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 - \left(\frac{-2}{(x^2+3)} + \frac{6}{(x^2+4)}\right)$

$$\Rightarrow \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} dx = \int \left\{1 + \frac{2}{(x^2+3)} - \frac{6}{(x^2+4)}\right\} dx$$

$$= \int \left\{1 + \frac{2}{x^2 + \left(\sqrt{3}\right)^2} - \frac{6}{x^2 + 2^2}\right\} dx$$

$$= x + 2\left(\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right) - 6\left(\frac{1}{2}\tan^{-1}\frac{x}{2}\right) + C$$

$$= x + \frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 3\tan^{-1}\frac{x}{2} + C$$

Hence, integration of $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = x + \frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 3\tan^{-1}\frac{x}{2} + C$ [2 Marks]

19. Integrate the rational function $\frac{2x}{(x^2+1)(x^2+3)}$ [6 Marks]

Solution:

We need to integrate $\frac{2x}{(x^2+1)(x^2+3)}$

Let $x^2 = t \Rightarrow 2x\,dx = dt$

$\therefore \int \frac{2x}{(x^2+1)(x^2+3)} dx = \int \frac{dt}{(t+1)(t+3)}$ ... (1) [1 Mark]

Let $\frac{1}{(t+1)(t+3)} = \frac{A}{(t+1)} + \frac{B}{(t+3)}$ [$\frac{1}{2}$ Mark]

$\Rightarrow 1 = A(t+3) + B(t+1)$

Equating the coefficients of t and constant, we get

$A + B = 0$ and $3A + B = 1$

On solving, we get

$A = \frac{1}{2}$ and $B = -\frac{1}{2}$ **[2 Marks]**

$\therefore \frac{1}{(t+1)(t+3)} = \frac{1}{2(t+1)} - \frac{1}{2(t+3)}$ **[$\frac{1}{2}$ Mark]**

$$\Rightarrow \int \frac{2x}{(x^2+1)(x^2+3)} dx = \int \left\{\frac{1}{2(t+1)} - \frac{1}{2(t+3)}\right\} dt$$

$$= \frac{1}{2}\log|t+1| - \frac{1}{2}\log|t+3| + C$$

$$= \frac{1}{2}\log\left|\frac{t+1}{t+3}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + C$$

Hence, integration of $\frac{2x}{(x^2+1)(x^2+3)} = \frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + C$ **[2 Marks]**

20. Integrate the rational function $\frac{1}{x(x^4-1)}$ **[6 Marks]**

Solution:

We need to integrate $\frac{1}{x(x^4-1)}$

Multiplying numerator and denominator by x^3, we get

$$\frac{1}{x(x^4-1)} = \frac{x^3}{x^4(x^4-1)}$$

$$\therefore \int \frac{1}{x(x^4-1)} dx = \int \frac{x^3}{x^4(x^4-1)} dx$$

Let $x^4 = t \Rightarrow 4x^3 dx = dt$ **[1 Mark]**

$$\therefore \int \frac{1}{x(x^4-1)} dx = \frac{1}{4}\int \frac{dt}{t(t-1)}$$

Let $\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{(t-1)}$

$\Rightarrow 1 = A(t-1) + Bt$ ... (1) [$\frac{1}{2}$ Mark]

Equating the coefficients of t and constant, we get

$A = -1$ and $B = 1$ [1 Mark]

$\Rightarrow \frac{1}{t(t+1)} = \frac{-1}{t} + \frac{1}{t-1}$ [$\frac{1}{2}$ Mark]

$$\Rightarrow \int \frac{1}{x(x^4-1)} dx = \frac{1}{4} \int \left\{ \frac{-1}{t} + \frac{1}{t-1} \right\} dt$$

$$= \frac{1}{4} [-\log|t| + \log|t-1|] + C$$

$$= \frac{1}{4} \log \left| \frac{t-1}{t} \right| + C$$

$$= \frac{1}{4} \log \left| \frac{x^4-1}{x^4} \right| + C$$

Hence, integration of $\frac{1}{x(x^4-1)} = \frac{1}{4} \log \left| \frac{x^4-1}{x^4} \right| + C$ [3 Marks]

21. Integrate the rational function $\frac{1}{(e^x-1)}$ [Hint: Put $e^x = t$] **[4 Marks]**

Solution:

We need to integrate $\frac{1}{(e^x-1)}$

Let $e^x = t \Rightarrow e^x dx = dt$ [$\frac{1}{2}$ Mark]

$$\Rightarrow \int \frac{1}{e^x-1} dx = \int \frac{1}{t-1} \times \frac{dt}{t} = \int \frac{1}{t(t-1)} dt$$

Let $\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{t-1}$

$\Rightarrow 1 = A(t-1) + Bt \ldots (1)$

Equating the coefficient of t and constant, we get

$A = -1$ and $B = 1$ [1 Mark]

$\therefore \frac{1}{t(t-1)} = \frac{-1}{t} + \frac{1}{t-1}$ [$\frac{1}{2}$ Mark]

$$\Rightarrow \int \frac{1}{t(t-1)} dt = \int \left\{ \frac{-1}{t} + \frac{1}{t-1} \right\} dt$$

$$= -\log|t| + \log|t-1| + C$$

$$= \log\left|\frac{t-1}{t}\right| + C$$

$$= \log\left|\frac{e^x - 1}{e^x}\right| + C$$

Hence, integration of $\frac{1}{(e^x-1)} = \log\left|\frac{e^x-1}{e^x}\right| + C$ [2 Marks]

22. $\int \frac{x\,dx}{(x-1)(x-2)}$ equals [4 Marks]

(A) $\log\left|\frac{(x-1)^2}{x-2}\right| + C$

(B) $\log\left|\frac{(x-2)^2}{x-1}\right| + C$

(C) $\log\left|\left(\frac{x-1}{x-2}\right)^2\right| + C$

(D) $\log|(x-1)(x-2)| + C$

Solution:

We need to find $\int \frac{x\,dx}{(x-1)(x-2)}$

Let $\frac{x}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$

$\Rightarrow x = A(x-2) + B(x-1) \dots (1)$ [$\frac{1}{2}$ Mark]

Equating the coefficient of x and constant, we get

$A = -1$, and $B = 2$ [1 Mark]

$\therefore \frac{x}{(x-1)(x-2)} = -\frac{1}{(x-1)} + \frac{2}{(x-2)}$ [$\frac{1}{2}$ Mark]

$$\Rightarrow \int \frac{x}{(x-1)(x-2)} dx = \int \left\{ \frac{-1}{(x-1)} + \frac{2}{(x-2)} \right\} dx$$

$$= -\log|x-1| + 2\log|x-2| + C$$

$$= \log\left|\frac{(x-2)^2}{x-1}\right| + C$$

Hence, the correct answer is B. [2 Marks]

23. $\int \frac{dx}{x(x^2+1)}$ equals **[4 Marks]**

(A) $\log|x| - \frac{1}{2}\log(x^2+1) + C$

(B) $\log|x| + \frac{1}{2}\log(x^2+1) + C$

(C) $-\log|x| + \frac{1}{2}\log(x^2+1) + C$

(D) $\frac{1}{2}\log|x| + \log(x^2+1) + C$

Solution:

We need to find $\int \frac{dx}{x(x^2+1)}$

Let $\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$\Rightarrow 1 = A(x^2+1) + (Bx+C)x$ [$\frac{1}{2}$ Mark]

Equating the coefficients of x^2, x, and constant term, we get

$A + B = 0$

$C = 0$

$A = 1$

On solving these equations, we get

$A = 1, B = -1$, and $C = 0$ **[1 Mark]**

$\therefore \frac{1}{x(x^2+1)} = \frac{1}{x} + \frac{-x}{x^2+1}$ **[$\frac{1}{2}$ Mark]**

$$\Rightarrow \int \frac{1}{x(x^2+1)} dx = \int \left\{\frac{1}{x} - \frac{x}{x^2+1}\right\} dx$$

$$= \log|x| - \frac{1}{2}\log|x^2+1| + C$$

Hence, the correct answer is A **[2 Marks]**

Exercise 7.6

1. Integrate the function $x \sin x$ **[2 Marks]**

Solution:

We need to integrate $x \sin x$

Let $I = \int x \sin x \, dx$

Taking x as first function and $\sin x$ as second function and integrating using by parts, we get

$I = x \int \sin x \, dx - \int \left\{\left(\frac{d}{dx}x\right) \int \sin x \, dx\right\} dx$

$[\because \int f(x)g(x)dx = f(x)\int g(x)dx - \int[f'(x)\int g(x)dx]dx]$ **[1 Mark]**

$$= x(-\cos x) - \int 1 \cdot (-\cos x)dx$$

$$= -x\cos x + \sin x + C$$

Hence, integration of $x \sin x = -x\cos x + \sin x + C$ **[1 Mark]**

2. Integrate the function $x \sin 3x$. **[2 Marks]**

Solution:

We need to integrate $x \sin 3x$

Let $I = \int x \sin 3x \, dx$

Taking x as first function and $\sin 3x$ as second function and integrating using by parts, we get

$I = x \int \sin 3x \, dx - \int \left\{ \left(\frac{d}{dx} x \right) \int \sin 3x dx \right\} dx$

$[\because \int f(x)g(x)dx = f(x)\int g(x)dx - \int [f'(x)\int g(x)dx]dx]$ **[1 Mark]**

$= x\left(\frac{-\cos 3x}{3}\right) - \int 1 \cdot \left(\frac{-\cos 3x}{3}\right) dx$

$= \frac{-x \cos 3x}{3} + \frac{1}{3}\int \cos 3x \, dx$

$= \frac{-x \cos 3x}{3} + \frac{1}{9}\sin 3x + C$

Hence, Integration of $x \sin 3x = \frac{-x \cos 3x}{3} + \frac{1}{9}\sin 3x + C$ **[1 Mark]**

3. Integrate the function $x^2 e^x$ **[4 Marks]**

Solution:

We need to integrate $x^2 e^x$

Let $I = \int x^2 e^x \, dx$

Taking x^2 as first function and e^x as second function and integrating using by parts, we get

$I = x^2 \int e^x dx - \int \left\{ \left(\frac{d}{dx} x^2 \right) \int e^x dx \right\} dx \, [\because \int f(x)g(x)dx = f(x)\int g(x)dx - \int [f'(x)\int g(x)dx] dx]$

[1 Mark]

$= x^2 e^x - \int 2x \cdot e^x dx$

$= x^2 e^x - 2\int x \cdot e^x \, dx$ **[1 Mark]**

Again, integrating using by parts, we get

$= x^2e^x - 2\left[x \cdot \int e^x dx - \int\left\{\left(\frac{d}{dx}x\right) \cdot \int e^x dx\right\} dx\right]$ **[1 Mark]**

$= x^2e^x - 2[xe^x - \int e^x\, dx]$

$= x^2e^x - 2[xe^x - e^x] + C$

$= x^2e^x - 2xe^x + 2e^x + C$

$= e^x(x^2 - 2x + 2) + C$

Hence, integration of $x^2e^x = e^x(x^2 - 2x + 2) + C$ **[1 Mark]**

4. Integrate the function $x \log x$ **[2 Marks]**

Solution:

We need to integrate $x \log x$

Let $I = \int x \log x\, dx$

Taking $\log x$ as first function and x as second function and integrating using by parts, we get,

$I = \log x \int x\, dx - \int\left\{\left(\frac{d}{dx}\log x\right)\int x\, dx\right\} dx$

$[\because \int f(x)g(x)dx = f(x)\int g(x)dx - \int[f'(x)\int g(x)dx]dx]$ **[1 Mark]**

$= \log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx$

$= \frac{x^2 \log x}{2} - \int \frac{x}{2} dx$

$= \frac{x^2 \log x}{2} - \frac{x^2}{4} + C$

Hence, integration of $x \log x = \frac{x^2 \log x}{2} - \frac{x^2}{4} + C$ **[1 Mark]**

5. Integrate the function $x \log 2x$ **[2 Marks]**

Solution:

We need to integrate $x\log 2x$

Let $I=\int x\log 2x\,dx$

Taking $\log 2x$ as first function and x as second function and integrating using by parts, we get,

$I=\log 2x\int x\,dx-\int\left\{\left(\frac{d}{dx}2\log x\right)\int x\,dx\right\}dx$

$[\because \int f(x)g(x)dx=f(x)\int g(x)dx-\int[f'(x)\int g(x)dx]dx]$ **[1 Mark]**

$=\log 2x\cdot\frac{x^2}{2}-\int\frac{2}{2x}\cdot\frac{x^2}{2}dx$

$=\frac{x^2\log 2x}{2}-\int\frac{x}{2}dx$

$=\frac{x^2\log 2x}{2}-\frac{x^2}{4}+C$

Hence, integration of $x\log 2x=\frac{x^2\log 2x}{2}-\frac{x^2}{4}+C$ **[1 Mark]**

6. Integrate the function $x^2\log x$ **[2 Marks]**

Solution:

We need to integrate $x^2\log x$

Let $I=\int x^2\log x\,dx$

Taking $\log x$ as first function and x^2 as second function and integrating using by parts, we get,

$I=\log x\int x^2\,dx-\int\left\{\left(\frac{d}{dx}\log x\right)\int x^2\,dx\right\}dx$

$[\because \int f(x)g(x)dx=f(x)\int g(x)dx-\int[f'(x)\int g(x)dx]dx]$ **[1 Mark]**

$=\log x\left(\frac{x^3}{3}\right)-\int\frac{1}{x}\cdot\frac{x^3}{3}dx$

$= \frac{x^3 \log x}{3} - \int \frac{x^2}{3} dx$

$= \frac{x^3 \log x}{3} - \frac{x^3}{9} + C$

Hence, integration of $x^2 \log x = \frac{x^3 \log x}{3} - \frac{x^3}{9} + C$ **[1 Mark]**

7. Integrate the function $x \sin^{-1} x$ **[4 Marks]**

Solution:

We need to integrate $x \sin^{-1} x$

Let $I = \int x \sin^{-1} x \, dx$

Taking $\sin^{-1} x$ as first function and x as second function and integrating using by parts, we get,

$I = \sin^{-1}x \int x \, dx - \int \left\{\left(\frac{d}{dx}\sin^{-1}x\right) \int x \, dx\right\} dx$ **[1 Mark]**

$[\because \int f(x)g(x)dx = f(x)\int g(x)dx - \int[f'(x)\int g(x)dx]dx]$

$= \sin^{-1}x \left(\frac{x^2}{2}\right) - \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} dx$

$= \frac{x^2 \sin^{-1}x}{2} + \frac{1}{2} \int \frac{-x^2}{\sqrt{1-x^2}} dx$

$= \frac{x^2 \sin^{-1}x}{2} + \frac{1}{2} \int \left\{\frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}\right\} dx$ **[1 Mark]**

$= \frac{x^2 \sin^{-1}x}{2} + \frac{1}{2} \int \left\{\sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}}\right\} dx$

$= \frac{x^2 \sin^{-1}x}{2} + \frac{1}{2} \left\{\int \sqrt{1-x^2} \, dx - \int \frac{1}{\sqrt{1-x^2}} dx\right\}$

$= \frac{x^2 \sin^{-1}x}{2} + \frac{1}{2} \left\{\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x - \sin^{-1}x\right\} + C$ **[1 Mark]**

$= \frac{x^2 \sin^{-1}x}{2} + \frac{x}{4}\sqrt{1-x^2} + \frac{1}{4}\sin^{-1}x - \frac{1}{2}\sin^{-1}x + C$

$= \frac{1}{4}(2x^2 - 1)\sin^{-1}x + \frac{x}{4}\sqrt{1-x^2} + C$

Hence, integration of $x\sin^{-1}x = \frac{1}{4}(2x^2-1)\sin^{-1}x + \frac{x}{4}\sqrt{1-x^2} + C$ **[1 Mark]**

8. Integrate the function $x\tan^{-1}x$ **[4 Marks]**

Solution:

We need to integrate $x\tan^{-1}x$

Let $I = \int x\tan^{-1}x\,dx$

Taking $\tan^{-1}x$ as first function and x as second function and integrating using by parts, we get,

$I = \tan^{-1}x\int x\,dx - \int\left\{\left(\frac{d}{dx}\tan^{-1}x\right)\int x\,dx\right\}dx$ **[1 Mark]**

$[\because \int f(x)g(x)dx = f(x)\int g(x)dx - \int[f'(x)\int g(x)dx]dx]$

$= \tan^{-1}x\left(\frac{x^2}{2}\right) - \int \frac{1}{1+x^2}\cdot\frac{x^2}{2}dx$

$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\int\frac{x^2}{1+x^2}dx$

$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\int\left(\frac{x^2+1}{1+x^2} - \frac{1}{1+x^2}\right)dx$ **[1 Mark]**

$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$

$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}(x - \tan^{-1}x) + C$ **[1 Mark]**

$= \frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1}x + C$

Hence, Integration of $x\tan^{-1}x = \frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1}x + C$ **[1 Mark]**

9. Integrate the function $x\cos^{-1}x$ **[4 Marks]**

Solution:

We need to integrate $x\cos^{-1}x$

Let $I=\int x\cos^{-1}x\,dx$

Taking $\cos^{-1}x$ as first function and x as second function and integrating using by parts, we get,

$I=\cos^{-1}x\int x\,dx-\int\left\{\left(\frac{d}{dx}\cos^{-1}x\right)\int x\,dx\right\}dx$ **[1 Mark]**

$[\because \int f(x)g(x)dx=f(x)\int g(x)dx-\int[f'(x)\int g(x)dx]dx]$

$=\cos^{-1}x\frac{x^2}{2}-\int\frac{-1}{\sqrt{1-x^2}}\cdot\frac{x^2}{2}dx$

$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\int\frac{1-x^2-1}{\sqrt{1-x^2}}dx$

$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\int\left\{\sqrt{1-x^2}+\left(\frac{-1}{\sqrt{1-x^2}}\right)\right\}dx$ **[1 Mark]**

$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\int\sqrt{1-x^2}\,dx-\frac{1}{2}\int\left(\frac{-1}{\sqrt{1-x^2}}\right)dx$

$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x\right]-\frac{1}{2}\cos^{-1}x+C$ **[1 Mark]**

$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\left[\frac{x}{2}\sqrt{1-x^2}-\frac{1}{2}\cos^{-1}x\right]-\frac{1}{2}\cos^{-1}x+C_1$

$=\frac{(2x^2-1)}{4}\cos^{-1}x-\frac{x}{4}\sqrt{1-x^2}+C_1$

Hence, integration of $x\cos^{-1}x=\frac{(2x^2-1)}{4}\cos^{-1}x-\frac{x}{4}\sqrt{1-x^2}+C_1$ **[1 Mark]**

10. Integrate the function $(\sin^{-1}x)^2$ **[4 Marks]**

Solution:

We need to integrate $(\sin^{-1}x)^2$

Let $I=\int(\sin^{-1}x)^2\cdot 1\,dx$

Taking $(\sin^{-1} x)^2$ as first function and 1 as second function and integrating using by parts, we get,

$I = (\sin^{-1}x)\int 1\,dx - \int\left\{\frac{d}{dx}(\sin^{-1}x)^2 \cdot \int 1 \cdot dx\right\}dx$ **[1 Mark]**

$[\because \int f(x)g(x)dx = f(x)\int g(x)dx - \int[f'(x)\int g(x)dx]dx]$

$$= (\sin^{-1}x)^2 \cdot x - \int \frac{2\sin^{-1}x}{\sqrt{1-x^2}} \cdot xdx$$

$$= x(\sin^{-1}x)^2 + \int \sin^{-1}x \cdot \left(\frac{-2x}{\sqrt{1-x^2}}\right)dx$$

$= x(\sin^{-1}x)^2 + \left[\sin^{-1}x \int \frac{-2x}{\sqrt{1-x^2}}dx - \int\left\{\left(\frac{d}{dx}\sin^{-1}x\right)\int \frac{-2x}{\sqrt{1-x^2}}dx\right\}dx\right]$

$= x(\sin^{-1}x)^2 + \left[\sin^{-1}x \cdot 2\sqrt{1-x^2} - \int \frac{1}{\sqrt{1-x^2}} \cdot 2\sqrt{1-x^2}dx\right]$ **[2 Marks]**

$$= x(\sin^{-1}x)^2 + 2\sqrt{1-x^2}\sin^{-1}x - \int 2\,dx$$

$$= x(\sin^{-1}x)^2 + 2\sqrt{1-x^2}\sin^{-1}x - 2x + C$$

Hence, integration of $(\sin^{-1} x)^2 = x(\sin^{-1}x)^2 + 2\sqrt{1-x^2}\sin^{-1}x - 2x + C$ **[1 Mark]**

11. Integrate the function $\frac{x\cos^{-1}x}{\sqrt{1-x^2}}$ **[4 Marks]**

Solution:

We need to integrate $\frac{x\cos^{-1}x}{\sqrt{1-x^2}}$

Let $I = \int \frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$

$$I = \frac{-1}{2}\int \frac{-2x}{\sqrt{1-x^2}} \cdot \cos^{-1}xdx$$

Taking $\cos^{-1} x$ as first function and $\left(\frac{-2x}{\sqrt{1-x^2}}\right)$ as second function and integrating using by parts, we get,

$I = \frac{-1}{2}\left[\cos^{-1}x \int \frac{-2x}{\sqrt{1-x^2}}dx - \int\left\{\left(\frac{d}{dx}\cos^{-1}x\right)\int \frac{-2x}{\sqrt{1-x^2}}dx\right\}dx\right]$ **[1 Mark]**

$[\because \int f(x)g(x)dx = f(x)\int g(x)dx - \int[f'(x)\int g(x)dx]dx]$

$= \frac{-1}{2}\left[\cos^{-1}x \cdot 2\sqrt{1-x^2} - \int \frac{-1}{\sqrt{1-x^2}} \cdot 2\sqrt{1-x^2}dx\right]$ **[1 Mark]**

$$= \frac{-1}{2}\left[2\sqrt{1-x^2}\cos^{-1}x + \int 2\,dx\right]$$

$$= \frac{-1}{2}\left[2\sqrt{1-x^2}\cos^{-1}x + 2x\right] + C$$

$$= -\left[\sqrt{1-x^2}\cos^{-1}x + x\right] + C$$

Hence, integration of $\frac{x\cos^{-1}x}{\sqrt{1-x^2}} = -\left[\sqrt{1-x^2}\cos^{-1}x + x\right] + C$ **[2 Marks]**

12. Integrate the function $x\sec^2 x$ **[2 Marks]**

Solution:

Let $I = \int x\sec^2 x dx$

Taking x as first function and $\sec^2 x$ as second function and integrating using by parts, we get,

$I = x\int \sec^2 x\,dx - \int\left\{\left\{\frac{d}{dx}x\right\}\int \sec^2 x\,dx\right\}dx$ **[1 Mark]**

$[\because \int f(x)g(x)dx = f(x)\int g(x)dx - \int[f'(x)\int g(x)dx]dx]$

$$= x\tan x - \int 1 \cdot \tan x\,dx$$

$$= x\tan x + \log|\cos x| + C$$

Hence, integration of $x\sec^2 x = x\tan x + \log|\cos x| + C$ **[1 Mark]**

13. Integrate the function $\tan^{-1} x$ **[2 Marks]**

Solution:

We need to integrate $\tan^{-1} x$

Let $I = \int 1 \cdot \tan^{-1} x dx$

Taking $\tan^{-1} x$ as first function and 1 as second function and integrating using by parts, we get,

$I = \tan^{-1}x \int 1\, dx - \int \left\{\left(\frac{d}{dx}\tan^{-1}x\right) \int 1 \cdot dx\right\} dx$ **[1 Mark]**

$[\because \int f(x)g(x)dx = f(x)\int g(x)dx - \int[f'(x)\int g(x)dx]dx]$

$= \tan^{-1}x \cdot x - \int \frac{1}{1+x^2} \cdot xdx$

$= x\tan^{-1}x - \frac{1}{2}\int \frac{2x}{1+x^2} dx$

$= x\tan^{-1}x - \frac{1}{2}\log|1+x^2| + C$

$= x\tan^{-1}x - \frac{1}{2}\log(1+x^2) + C$

Hence, integration of $\tan^{-1} x = x\tan^{-1}x - \frac{1}{2}\log(1+x^2) + C$ **[1 Mark]**

14. Integrate the function $x\,(\log x)^2$ **[4 Marks]**

Solution:

We need to integrate $x\,(\log x)^2$

$I = \int x\,(\log x)^2\, dx$

Taking $(\log x)^2$ as first function and x as second function and integrating using by parts, we get

$I = (\log x)^2 \int xdx - \int \left[\left\{\left(\frac{d}{dx}\log x\right)^2\right\} \int xdx\right] dx$ **[1 Mark]**

$[\because \int f(x)g(x)dx = f(x)\int g(x)dx - \int[f'(x)\int g(x)dx]dx]$

$= \frac{x^2}{2}(\log x)^2 - \left[\int 2\log x \cdot \frac{1}{x} \cdot \frac{x^2}{2} dx\right]$

$= \frac{x^2}{2}(\log x)^2 - \int x \log x \, dx$ **[1 Mark]**

Again, integrating using by parts, we get

$I = \frac{x^2}{2}(\log x)^2 - \left[\log x \int x\, dx - \int \left\{\left(\frac{d}{dx}\log x\right) \int x\, dx\right\} dx\right]$ **[1 Mark]**

$[\because \int f(x)g(x)dx = f(x)\int g(x)dx - \int[f'(x)\int g(x)dx]dx]$

$= \frac{x^2}{2}(\log x)^2 - \left[\frac{x^2}{2} - \log x - \int \frac{1}{x} \cdot \frac{x^2}{2} dx\right]$

$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{1}{2}\int x\, dx$

$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{x^2}{4} + C$

Hence, integration of $x\,(\log x)^2 = \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{x^2}{4} + C$ **[1 Mark]**

15. Integrate the function $(x^2 + 1)\log x$ **[6 Marks]**

Solution:

We need to integrate $(x^2 + 1)\log x$

Let $I = \int (x^2 + 1)\log x\, dx = \int x^2 \log x\, dx + \int \log x\, dx$

$I = I_1 + I_2 \dots (1)$ **[1 Mark]**

$I_1 = \int x^2 \log x\, dx$ and $I_2 = \int \log x\, dx$

$I_1 = \int x^2 \log x\, dx$

Taking $\log x$ as fist function and x^2 as second function and integrating using by parts, we get

$I_1 = \log x - \int x^2\, dx - \int \left\{\left(\frac{d}{dx}\log x\right) \int x^2\, dx\right\} dx$ **[1 Mark]**

$[\because \int f(x)g(x)dx = f(x)\int g(x)dx - \int[f'(x)\int g(x)dx]dx]$

$= \log x \cdot \frac{x^3}{3} - \int \frac{1}{x} \cdot \frac{x^3}{3} dx$

$= \frac{x^3}{3}\log x - \frac{1}{3}\left(\int x^2\, dx\right)$

$= \frac{x^3}{3}\log x - \frac{x^3}{9} + C_1 \ldots (2)$ **[1 Mark]**

$I_2 = \int \log x\, dx$

Taking $\log x$ as first function and 1 as second function and integrating using by parts, we get

$I_2 = \log x \int 1 \cdot dx - \int\left\{\left(\frac{d}{dx}\log x\right)\int 1 \cdot dx\right\}$ **[1 Mark]**

$[\because \int f(x)g(x)dx = f(x)\int g(x)dx - \int[f'(x)\int g(x)dx]dx]$

$= \log x \cdot x - \int \frac{1}{x} \cdot x dx$

$= x\log x - \int 1\, dx$

$= x\log x - x + C_2 \ldots (3)$ **[1 Mark]**

Using equations (2) and (3) in (1), we get

$I = \frac{x^3}{3}\log x - \frac{x^3}{9} + C_1 + x\log x - x + C_2$

$= \frac{x^3}{3}\log x - \frac{x^3}{9} + x\log x - x + (C_1 + C_2)$

$= \left(\frac{x^3}{3} + x\right)\log x - \frac{x^3}{9} - x + C$

Hence, integration of $(x^2+1)\log x = \left(\frac{x^3}{3} + x\right)\log x - \frac{x^3}{9} - x + C$ **[1 Mark]**

16. Integrate the function $e^x(\sin x + \cos x)$ **[2 Marks]**

Solution:

We need to integrate $e^x(\sin x + \cos x)$

Let $I = \int e^x(\sin x + \cos x)\, dx$

Again, let $f(x) = \sin x$

Hence, $f'(x) = \cos x$

$I = \int e^x\{f(x) + f'(x)\}\, dx$ **[1 Mark]**

As we known that, $\int e^x\{f(x) + f'(x)\}dx = e^x f(x) + C$

$\therefore I = e^x \sin x + C$

Hence, integration of $e^x(\sin x + \cos x) = e^x \sin x + C$ **[1 Mark]**

17. Integrate the function $\frac{xe^x}{(1+x)^2}$ **[2 Marks]**

Solution:

We need to integrate $\frac{xe^x}{(1+x)^2}$

Let $I = \int \frac{xe^x}{(1+x)^2} dx = \int e^x \left\{\frac{x}{(1+x)^2}\right\} dx$

$$= \int e^x \left\{\frac{1 + x - 1}{(1 + x)^2}\right\} dx$$

$$= \int e^x \left\{\frac{1}{1 + x} - \frac{1}{(1 + x)^2}\right\} dx$$

Let $f(x) = \frac{1}{1+x}$

Hence, $f'(x) = \frac{-1}{(1+x)^2}$

$\Rightarrow \int \frac{xe^x}{(1+x)^2} dx = \int e^x\{f(x) + f'(x)\}\, dx$ **[1 Mark]**

As we known that, $\int e^x\{f(x) + f'(x)\}\, dx = e^x f(x) + C$

$$\therefore \int \frac{xe^x}{(1 + x)^2} dx = \frac{e^x}{1 + x} + C$$

Hence, integration of $\frac{xe^x}{(1+x)^2} = \int \frac{xe^x}{(1+x)^2} dx = \frac{e^x}{1+x} + C$ **[1 Mark]**

18. Integrate the function $e^x\left(\frac{1+\sin x}{1+\cos x}\right)$ **[4 Marks]**

Solution:

We need to integrate $e^x\left(\frac{1+\sin x}{1+\cos x}\right)$

$= e^x\left(\frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)$

$= \frac{e^x\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}}$

$= \frac{1}{2}e^x\cdot\left(\frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\cos\frac{x}{2}}\right)^2$

$= \frac{1}{2}e^x\left[\tan\frac{x}{2}+1\right]^2$

$= \frac{1}{2}e^x\left(1+\tan\frac{x}{2}\right)^2$

$= \frac{1}{2}e^x\left[1+\tan^2\frac{x}{2}+2\tan\frac{x}{2}\right]$

$= \frac{1}{2}e^x\left[\sec^2\frac{x}{2}+2\tan\frac{x}{2}\right]$

Hence, $\int\frac{e^x(1+\sin x)}{(1+\cos x)}dx = \int e^x\left[\frac{1}{2}\sec^2\frac{x}{2}+\tan\frac{x}{2}\right]dx$ ... (1) **[2 Marks]**

Let $\tan\frac{x}{2} = f(x)$

Hence, $f'(x) = \frac{1}{2}\sec^2\frac{x}{2}$

As we known that, $\int e^x\{f(x)+f'(x)\}\,dx = e^x f(x) + C$ **[1 Mark]**

From equation (1) we get

$\int\frac{e^x(1+\sin x)}{(1+\cos x)}dx = e^x\tan\frac{x}{2} + C$

Hence, integration of $e^x\left(\frac{1+\sin x}{1+\cos x}\right) = e^x\tan\frac{x}{2} + C$ **[1 Mark]**

19. Integrate the function $e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)$ **[2 Marks]**

Solution:

We need to integrate $e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)$

Let $I=\int e^x\left[\frac{1}{x}-\frac{1}{x^2}\right]dx$

Also, let $\frac{1}{x}=f(x)$

Hence, $f'(x)=\frac{-1}{x^2}$

As we known that, $\int e^x\{f(x)+f'(x)\}\,dx=e^xf(x)+C$ **[1 Mark]**

$\therefore I=\dfrac{e^x}{x}+C$

Hence, integration of $e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)=\frac{e^x}{x}+C$ **[1 Mark]**

20. Integrate the function $\frac{(x-3)e^x}{(x-1)^3}$ **[2 Marks]**

Solution:

We need to integrate $\frac{(x-3)e^x}{(x-1)^3}$

$\int e^x\left\{\frac{x-3}{(x-1)^3}\right\}dx=\int e^x\left\{\frac{x-1-2}{(x-1)^3}\right\}dx$

$=\int e^x\left\{\frac{1}{(x-1)^2}-\frac{2}{(x-1)^3}\right\}dx$

$f(x)=\frac{1}{(x-1)^2}$

Hence, $f'(x)=\frac{-2}{(x-1)^3}$ **[1 Mark]**

As we known that, $\int e^x\{f(x)+f'(x)\}\,dx=e^xf(x)+C$

$\therefore \int e^x\left\{\frac{(x-3)}{(x-1)^2}\right\}dx=\frac{e^x}{(x-1)^2}+C$

Hence, integration of $\frac{(x-3)e^x}{(x-1)^3} = \frac{e^x}{(x-1)^2} + C$ **[1 Mark]**

21. Integrate the function $e^{2x}\sin x$ **[4 Marks]**

Solution:

We need to integrate $e^{2x}\sin x$

Let $I = \int e^{2x}\sin x\,dx$ ... (1)

Integrating using by parts, we get

$I = \sin x\int e^{2x}\,dx - \int\left\{\left(\frac{d}{dx}\sin x\right)\int e^{2x}\,dx\right\}dx$ **[1 Mark]**

$\Rightarrow I = \sin x \cdot \frac{e^{2x}}{2} - \int \cos x \cdot \frac{e^{2x}}{2}dx$

$\Rightarrow I = \frac{e^{2x}\sin x}{2} - \frac{1}{2}\int e^{2x}\cos x\,dx$

Again, integrating using by parts, we get

$I = \frac{e^{2x}\cdot\sin x}{2} - \frac{1}{2}\left[\cos x\int e^{2x}\,dx - \int\left\{\left(\frac{d}{dx}\cos x\right)\int e^{2x}\,dx\right\}dx\right]$ **[1 Mark]**

$[\because \int f(x)g(x)dx = f(x)\int g(x)dx - \int[f'(x)\int g(x)dx]dx]$

$\Rightarrow I = \frac{e^{2x}\sin x}{2} - \frac{1}{2}\left[\cos x\cdot\frac{e^{2x}}{2} - \int(-\sin x)\frac{e^{2x}}{2}dx\right]$

$\Rightarrow I = \frac{e^{2x}\cdot\sin x}{2} - \frac{1}{2}\left[\frac{e^{2x}\cos x}{2} + \frac{1}{2}\int e^{2x}\sin x\,dx\right]$

$\Rightarrow I = \frac{e^{2x}\sin x}{2} - \frac{e^{2x}\cos x}{4} - \frac{1}{4}I$ [From (1)]

$\Rightarrow I + \frac{1}{4}I = \frac{e^{2x}\sin x}{2} - \frac{e^{2x}\cos x}{4}$

$\Rightarrow \frac{5}{4}I = \frac{e^{2x}\sin x}{2} - \frac{e^{2x}\cos x}{4}$

$\Rightarrow I = \frac{4}{5}\left[\frac{e^{2x}\sin x}{2} - \frac{e^{2x}\cos x}{4}\right] + C$

$\Rightarrow I = \frac{e^{2x}}{5}[2\sin x - \cos x] + C$

Hence, integration of $e^{2x}\sin x = \frac{e^{2x}}{5}[2\sin x - \cos x] + C$ **[2 Marks]**

22. Integrate the function $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ **[4 Marks]**

Solution:

We need to integrate $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Let $x = \tan\theta$

Hence, $dx = \sec^2\theta \, d\theta$ **[1 Mark]**

$\therefore \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$

$\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \int 2\theta \cdot \sec^2\theta d\theta = 2\int \theta \cdot \sec^2\theta d\theta$ **[1 Mark]**

Integrating using by parts, we get

$\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = 2\left[\theta \cdot \int \sec^2\theta d\theta - \int\left\{\left(\frac{d}{d\theta}\theta\right)\int \sec^2\theta d\theta\right\}d\theta\right]$ **[1 Mark]**

$[\because \int f(x)g(x)dx = f(x)\int g(x)dx - \int[f'(x)\int g(x)dx]dx]$

$= 2\left[\theta \cdot \tan\theta - \int \tan\theta \, d\theta\right]$

$= 2[\theta\tan\theta + \log|cos\theta|] + C$

$= 2\left[x\tan\theta + \log\left|\frac{1}{\sqrt{1+x^2}}\right|\right] + C$

$= 2x\tan^{-1}x + \log(1+x^2)^{\frac{1}{2}} + C$

$= 2x\tan^{-1}x - \log(1+x^2) + C$

Hence, integration of $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2x\tan^{-1}x - \log(1+x^2) + C$ **[1 Mark]**

23. $\int x^2 e^{x^3} dx$ equals **[2 Marks]**

(A) $\frac{1}{3}e^{x^3}+C$

(B) $\frac{1}{3}e^{x^2}+C$

(C) $\frac{1}{2}e^{x^3}+C$

(D) $\frac{1}{2}e^{x^2}+C$

Solution:

We need to integrate $\int x^2 e^{x^3} dx$

Let $I=\int x^2 e^{x^3} dx$

Also, let $x^3=t$

Hence, $3x^2dx=dt$

$\Rightarrow I=\frac{1}{3}\int e^t\,dt$ **[1 Mark]**

$=\frac{1}{3}(e^t)+C$

$=\frac{1}{3}e^{x^3}+C$

Hence, the correct Answer is A **[1 Mark]**

24. $\int e^x \sec x(1+\tan x)dx$ equals **[2 Marks]**

(A) $e^x\cos x+C$

(B) $e^x\sec x+C$

(C) $e^x\sin x+C$

(D) $e^x\tan x+C$

Solution:

We need to integrate $\int e^x \sec x(1+\tan x)dx$

Let $I = \int e^x \sec x(1+\tan x)dx = \int e^x (\sec x + \sec x \tan x)dx$

Also, let $\sec x = f(x)$

Hence, $f'(x) = \sec x \tan x$ **[1 Mark]**

As we known that, $\int e^x \{f(x) + f'(x)\}dx = e^x f(x) + C$

$\therefore I = e^x \sec x + C$

Hence, the correct Answer is B **[1 Mark]**

Exercise 7.7

1. Integrate the function $\sqrt{4-x^2}$ **[2 Marks]**

Solution:

We need to integrate $\sqrt{4-x^2}$

Let $I = \int \sqrt{4-x^2}\,dx = \int \sqrt{(2)^2-(x)^2}dx$

As we know that, $\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$ **[1 Mark]**

Here, $a = 2, x = x$

$\therefore I = \frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2} + C$

$= \frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2} + C$

Hence, integration of $\sqrt{4-x^2} = \frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2} + C$ **[1 Mark]**

2. Integrate the function $\sqrt{1-4x^2}$ **[2 Marks]**

Solution:

We need to integrate $\sqrt{1-4x^2}$

Let $I=\int\sqrt{1-4x^2}\,dx=\int\sqrt{(1)^2-(2x)^2}dx$

Let $2x=t\Rightarrow 2dx=dt$

$\therefore I=\frac{1}{2}\int\sqrt{(1)^2-(t)^2}dt$ **[1 Mark]**

As we know that, $\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

Here, $a=1, x=t$

$\Rightarrow I=\frac{1}{2}\left[\frac{t}{2}\sqrt{1-t^2}+\frac{1}{2}\sin^{-1}t\right]+C$

$=\frac{t}{4}\sqrt{1-t^2}+\frac{1}{4}\sin^{-1}t+C$

$=\frac{2x}{4}\sqrt{1-4x^2}+\frac{1}{4}\sin^{-1}2x+C$ [substituting t]

$=\frac{x}{2}\sqrt{1-4x^2}+\frac{1}{4}\sin^{-1}2x+C$

Hence, integration of $\sqrt{1-4x^2}=\frac{x}{2}\sqrt{1-4x^2}+\frac{1}{4}\sin^{-1}2x+C$ **[1 Mark]**

3. Integrate the function $\sqrt{x^2+4x+6}$ **[2 Marks]**

Solution:

We need to integrate $\sqrt{x^2+4x+6}$

Let $I=\int\sqrt{x^2+4x+6}\,dx$

$=\int\sqrt{x^2+4x+4+2}\,dx$

$=\int\sqrt{(x^2+4x+4)+2}\,dx$

$=\int\sqrt{(x+2)^2+\left(\sqrt{2}\right)^2}dx$ **[1 Mark]**

As we know that, $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$

Here, $a = \sqrt{2}, x = x+2$

$\therefore I = \dfrac{(x+2)}{2}\sqrt{x^2+4x+6} + \dfrac{2}{2}\log\left|(x+2)+\sqrt{x^2+4x+6}\right| + C$

$= \frac{(x+2)}{2}\sqrt{x^2+4x+6} + \log\left|(x+2)+\sqrt{x^2+4x+6}\right| + C$ **[1 Mark]**

Hence, integration of $\sqrt{x^2+4x+6} = \frac{(x+2)}{2}\sqrt{x^2+4x+6} + \log\left|(x+2)+\sqrt{x^2+4x+6}\right| + C$

4. Integrate the function $\sqrt{x^2+4x+1}$ **[2 Marks]**

Solution:

We need to integrate $\sqrt{x^2+4x+1}$

Let $I = \int \sqrt{x^2+4x+1}\,dx$

$= \displaystyle\int \sqrt{(x^2+4x+4)-3}\,dx$

$= \int \sqrt{(x+2)^2 - \left(\sqrt{3}\right)^2}dx$ **[1 Mark]**

As we know that, $\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$

Here, $a = \sqrt{3}, x = x+2$

$\therefore I = \frac{(x+2)}{2}\sqrt{x^2+4x+1} - \frac{3}{2}\log\left|(x+2)+\sqrt{x^2+4x+1}\right| + C$ **[1 Mark]**

Hence, integration of $\sqrt{x^2+4x+1} = \frac{(x+2)}{2}\sqrt{x^2+4x+1} - \frac{3}{2}\log\left|(x+2)+\sqrt{x^2+4x+1}\right| + C$

5. Integrate the function $\sqrt{1-4x-x^2}$ **[2 Marks]**

Solution:

We need to integrate $\sqrt{1-4x-x^2}$

Let $I=\int\sqrt{1-4x-x^2}dx$

$=\int\sqrt{1-(x^2+4x+4-4)}\,dx$

$=\int\sqrt{1+4-(x+2)^2}\,dx$

$=\int\sqrt{\left(\sqrt{5}\right)^2-(x+2)^2\,dx}$ **[1 Mark]**

As we know that, $\int\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

Here, $a=\sqrt{5}, x=x+2$

$\therefore I=\frac{(x+2)}{2}\sqrt{1-4x-x^2}+\frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right)+C$

Hence, integration of $\sqrt{1-4x-x^2}=\frac{(x+2)}{2}\sqrt{1-4x-x^2}+\frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right)+C$ **[1 Mark]**

6. Integrate the function $\sqrt{x^2+4x-5}$ **[2 Marks]**

Solution:

We need to integrate $\sqrt{x^2+4x-5}$

Let $I=\int\sqrt{x^2+4x-5}\,dx$

$=\int\sqrt{(x^2+4x+4)-9}\,dx$

$=\int\sqrt{(x+2)^2-(3)^2}\,dx$ **[1 Mark]**

As we know that, $\int\sqrt{x^2-a^2}dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

Here, $a=3, x=x+2$

$\therefore I = \frac{(x+2)}{2}\sqrt{x^2+4x-5} - \frac{9}{2}\log\left|(x+2)+\sqrt{x^2+4x-5}\right| + C$ **[1 Mark]**

Hence, integration of $\sqrt{x^2+4x-5} = \frac{(x+2)}{2}\sqrt{x^2+4x-5} - \frac{9}{2}\log\left|(x+2)+\sqrt{x^2+4x-5}\right| + C$

7. Integrate the function $\sqrt{1+3x-x^2}$ **[2 Marks]**

Solution:

We need to integrate $\sqrt{1+3x-x^2}$

Let $I = \int \sqrt{1+3x-x^2}\ dx$

$= \int \sqrt{1-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)}\,dx$

$= \int \sqrt{\left(1+\frac{9}{4}\right)-\left(x-\frac{3}{2}\right)^2}\ dx$

$= \int \sqrt{\left(\frac{\sqrt{13}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}\ dx$ **[1 Mark]**

As we know that, $\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

Here, $a = \frac{\sqrt{13}}{2}, x = x-\frac{3}{2}$

$\therefore I = \frac{x-\frac{3}{2}}{2}\sqrt{1+3x-x^2}+\frac{13}{4\times 2}\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{13}}{2}}\right)+C$

$= \frac{2x-3}{4}\sqrt{1+3x-x^2}+\frac{13}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right)+C$

Hence, integration of $\sqrt{1+3x-x^2} = \frac{2x-3}{4}\sqrt{1+3x-x^2}+\frac{13}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right)+C$ **[1 Mark]**

8. Integrate the function $\sqrt{x^2+3x}$ **[2 Marks]**

Solution:

We need to integrate $\sqrt{x^2+3x}$

Let $I=\int\sqrt{x^2+3x}\,dx$

$=\int\sqrt{x^2+3x+\frac{9}{4}-\frac{9}{4}}dx$

$=\int\sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2}\,dx$ **[1 Mark]**

As we know that, $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

Here, $a=\frac{3}{2}, x=x+\frac{3}{2}$

$\therefore I=\frac{\left(x+\frac{3}{2}\right)}{2}\sqrt{x^2+3x}-\frac{\frac{9}{4}}{2}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x}\right|+C$

$=\frac{(2x+3)}{4}\sqrt{x^2+3x}-\frac{9}{8}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x}\right|+C$

Hence, integration of $\sqrt{x^2+3x}=\frac{(2x+3)}{4}\sqrt{x^2+3x}-\frac{9}{8}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x}\right|+C$ **[1 Mark]**

9. Integrate the function $\sqrt{1+\frac{x^2}{9}}$ **[2 Marks]**

Solution:

We need to integrate $\sqrt{1+\frac{x^2}{9}}$

Let $I=\int\sqrt{1+\frac{x^2}{9}}dx=\frac{1}{3}\int\sqrt{9+x^2}\,dx=\frac{1}{3}\int\sqrt{(3)^2+x^2}\,\,dx$ **[1 Mark]**

As we know that, $\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$

Here, $a=3, x=x$

$\therefore I=\frac{1}{3}\left[\frac{x}{2}\sqrt{x^2+9}+\frac{9}{2}\log\left|x+\sqrt{x^2+9}\right|\right]+C$

$=\frac{x}{6}\sqrt{x^2+9}+\frac{3}{2}\log\left|x+\sqrt{x^2+9}\right|+C$

Hence, integration of $\sqrt{1+\frac{x^2}{9}}=\frac{x}{6}\sqrt{x^2+9}+\frac{3}{2}\log\left|x+\sqrt{x^2+9}\right|+C$ **[1 Mark]**

10. $\int\sqrt{1+x^2}\,dx$ is equal to **[1 Mark]**

(A) $\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log\left|x+\sqrt{1+x^2}\right|+C$

(B) $\frac{2}{3}(1+x^2)^{\frac{3}{2}}+C$

(C) $\frac{2}{3}x(1+x^2)^{\frac{3}{2}}+C$

(D) $\frac{x^2}{2}\sqrt{1+x^2}+\frac{1}{2}x^2\log\left|x+\sqrt{1+x^2}\right|+C$

Solution:

We need to integrate $\int\sqrt{1+x^2}\,dx$

As we know that, $\int\sqrt{a^2+x^2}\,dx=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$ **[$\frac{1}{2}$ Marks]**

Here, $a=1$ and $x=x$,

$\therefore \int\sqrt{1+x^2}\,dx=\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log\left|x+\sqrt{1+x^2}\right|+C$

Hence, the correct Answer is (A). **[$\frac{1}{2}$ Marks]**

11. $\int\sqrt{x^2-8x+7}\,dx$ is equal to **[2 Marks]**

(A) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}+9\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

(B) $\frac{1}{2}(x+4)\sqrt{x^2-8x+7}+9\log\left|x+4+\sqrt{x^2-8x+7}\right|+C$

(C) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-3\sqrt{2}\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

(D) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-\frac{9}{2}\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

Solution:

We need to integrate $\int \sqrt{x^2-8x+7}\,dx$

Let $I=\int \sqrt{x^2-8x+7}\,dx$

$=\int \sqrt{(x^2-8x+16)-9}\,dx$

$=\int \sqrt{(x-4)^2-(3)^2}\,dx$ **[1 Mark]**

As we know that, $\int \sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

Here, $a=3, x=x-4$

$\therefore I=\frac{(x-4)}{2}\sqrt{x^2-8x+7}-\frac{9}{2}\log\left|(x-4)+\sqrt{x^2-8x+7}\right|+C$

Hence, the correct answer is (D) **[1 Mark]**

Exercise 7.8

1. Evaluate the definite integral $\int_a^b x\,dx$ as limit of sum. **[4 Marks]**

Solution:

We know that,

$\int_a^b f(x)\,dx=(b-a)\lim_{n\to\infty}\frac{1}{n}[f(a)+f(a+h)+\cdots+f(a+(n-1)h)]$, where $h=\frac{b-a}{n}$

Here we have, $a=a, b=b$ and $f(x)=x$ **[1 Mark]**

$\therefore \int_a^b x dx = (b-a) \lim_{n\to\infty} \frac{1}{n}[a + (a+h) \dots (a+2h) \dots a + (n-1)h]$

$= (b-a) \lim_{n\to\infty} \frac{1}{n}\left[\left(\underbrace{a + a + a + \cdots + a}_{n \text{ times}}\right) + (h + 2h + 3h + \cdots + (n-1)h)\right]$ **[1 Mark]**

$= (b-a) \lim_{n\to\infty} \frac{1}{n}\left[na + h\left(1 + 2 + 3 + \cdots + (n-1)\right)\right]$

$= (b-a) \lim_{n\to\infty} \frac{1}{n}\left[na + h\left\{\frac{(n-1)(n)}{2}\right\}\right]$ **[1 Mark]**

$= (b-a) \lim_{n\to\infty} \frac{1}{n}\left[na + \frac{n(n-1)h}{2}\right]$

$= (b-a) \lim_{n\to\infty} \frac{n}{n}\left[a + \frac{(n-1)h}{2}\right]$

$= (b-a) \lim_{n\to\infty} \left[a + \frac{(n-1)h}{2}\right]$

$= (b-a) \lim_{n\to\infty} \left[a + \frac{(n-1)(b-a)}{2n}\right]$

$= (b-a) \lim_{n\to\infty} \left[a + \frac{\left(1-\frac{1}{n}\right)(b-a)}{2}\right]$

$= (b-a)\left[a + \frac{(b-a)}{2}\right]$

$= (b-a)\left[\frac{2a+b-a}{2}\right]$

$= \frac{(b-a)(b+a)}{2}$

$= \frac{1}{2}(b^2 - a^2)$ **[1 Mark]**

2. Evaluate the definite integral $\int_0^5 (x+1)dx$ as limit of sum. **[4 Marks]**

Solution:

Let $I = \int_0^5 (x+1)dx$

We know that,

$\int_a^b f(x)dx = (b-a) \lim_{n\to\infty} \frac{1}{n}[f(a) + f(a+h) \dots f(a+(n-1)h)]$, where $h = \frac{b-a}{n}$

Here we have, $a = 0, b = 5$ and $f(x) = (x + 1)$

$\Rightarrow h = \frac{5-0}{n} = \frac{5}{n}$ **[1 Mark]**

$\therefore \int_0^5 (x+1)dx = (5-0) \lim_{n\to\infty} \frac{1}{n}\left[f(0) + f\left(\frac{5}{n}\right) + \cdots + f\left((n-1)\frac{5}{n}\right)\right]$

$= 5 \lim_{n\to\infty} \frac{1}{n}\left[1 + \left(\frac{5}{n} + 1\right) + \ldots \left\{1 + \left(\frac{5(n-1)}{n}\right)\right\}\right]$

$= 5 \lim_{n\to\infty} \frac{1}{n}\left[\left(1 + \underset{n \text{ times}}{1 + 1} \ \ldots 1\right) + \left[\frac{5}{n} + 2 \cdot \frac{5}{n} + 3 \cdot \frac{5}{n} + \cdots (n-1)\frac{5}{n}\right]\right]$ **[1 Mark]**

$= 5 \lim_{n\to\infty} \frac{1}{n}\left[n + \frac{5}{n}\{1 + 2 + 3 \ldots (n-1)\}\right]$

$= 5 \lim_{n\to\infty} \frac{1}{n}\left[n + \frac{5}{n} \cdot \frac{(n-1)n}{2}\right]$ **[1 Mark]**

$= 5 \lim_{n\to\infty} \frac{1}{n}\left[n + \frac{5(n-1)}{2}\right]$

$= 5 \lim_{n\to\infty} \left[1 + \frac{5}{2}\left(1 - \frac{1}{n}\right)\right]$

$= 5\left[1 + \frac{5}{2}\right]$

$= 5\left[\frac{7}{2}\right]$

$= \frac{35}{2}$ **[1 Mark]**

3. Evaluate the definite integral $\int_2^3 x^2 dx$ as limit of sum. **[4 Marks]**

Solution:

We know that,

$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty} \frac{1}{n}[f(a) + f(a+h) + f(a+2h) \ldots f\{a + (n-1)h\}]$, where $h = \frac{b-a}{n}$

Here we have, $a = 2, b = 3$ and $f(x) = x^2$

$\Rightarrow h = \frac{3-2}{n} = \frac{1}{n}$ **[1 Mark]**

$\therefore \int_2^3 x^2\,dx = (3-2)\lim_{n\to\infty}\frac{1}{n}\left[f(2)+f\left(2+\frac{1}{n}\right)+f\left(2+\frac{2}{n}\right)\ldots f\left\{2+(n-1)\frac{1}{n}\right\}\right]$

$= \lim_{n\to\infty}\frac{1}{n}\left[(2)^2+\left(2+\frac{1}{n}\right)^2+\left(2+\frac{2}{n}\right)^2+\cdots\left(2+\frac{(n-1)}{n}\right)^2\right]$

$= \lim_{n\to\infty}\frac{1}{n}\left[2^2+\left\{2^2+\left(\frac{1}{n}\right)^2+2\cdot 2\cdot\frac{1}{n}\right\}+\cdots+\left\{(2)^2+\frac{(n-1)^2}{n^2}+2\cdot 2\cdot\frac{(n-1)}{n}\right\}\right]$

$= \lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{2^2+\cdots+2^2}_{n\text{ times}}\right)+\left\{\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\cdots+\left(\frac{n-1}{n}\right)^2\right\}+2\cdot 2\cdot\left\{\frac{1}{n}+\frac{2}{n}+\frac{3}{n}+\cdots+\frac{(n-1)}{n}\right\}\right]$

$= \lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{1}{n^2}\{1^2+2^2+3^2\ldots+(n-1)^2\}+\frac{4}{n}\{1+2+\cdots+(n-1)\}\right]$ **[1 Mark]**

$= \lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{1}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\}+\frac{4}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$

$= \lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{n\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{6}+\frac{4n-4}{2}\right]$ **[1 Mark]**

$= \lim_{n\to\infty}\left[4+\frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+2-\frac{2}{n}\right]$

$= 4+\frac{2}{6}+2$

$= \frac{19}{3}$ **[1 Mark]**

4. Evaluate the definite integral $\int_1^4 (x^2 - x)dx$ as limit of sum. **[6 Marks]**

Solution:

Let $I = \int_1^4 (x^2 - x)dx$

$= \int_1^4 x^2 dx - \int_1^4 x\,dx$

Let $I = I_1 - I_2$ ... (i)

where $I_1 = \int_1^4 x^2 dx$ and $I_2 = \int_1^4 x\,dx$

We know that,

$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a)+f(a+h)+f(a+(n-1)h)]$, where $h=\frac{b-a}{n}$

Here, $I_1 = \int_1^4 x^2 dx$,

$a=1, b=4$ and $f(x)=x^2$

$\therefore h = \frac{4-1}{n} = \frac{3}{n}$ **[1 Mark]**

$$I_1 = \int_1^4 x^2\,dx = (4-1)\lim_{n\to\infty}\frac{1}{n}[f(1)+f(1+h)+\cdots+f(1+(n-1)h)]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[1^2+\left(1+\frac{3}{n}\right)^2+\left(1+2\cdot\frac{3}{n}\right)^2+\cdots\left(1+\frac{(n-1)^3}{n}\right)^2\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[1^2+\left\{1^2+\left(\frac{3}{n}\right)^2+2\cdot\frac{3}{n}\right\}+\cdots+\left\{1^2+\left(\frac{(n-1)^3}{n}\right)^2+\frac{2\cdot(n-1)\cdot 3}{n}\right\}\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{1^2+\cdots+1^2}_{n\text{ times}}\right)+\left(\frac{3}{n}\right)^2\{1^2+2^2+\cdots+(n-1)^2\}+2\cdot\frac{3}{n}\{1+2+\cdots+(n-1)\}\right]$$ **[1 Mark]**

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{9}{n^2}\left\{\frac{(n-1)(n)(2n-1)}{6}\right\}+\frac{6}{n}\left\{\frac{(n-1)(n)}{2}\right\}\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{9n}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+\frac{6n-6}{2}\right]$$

$$= 3\lim_{n\to\infty}\left[1+\frac{9}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+3-\frac{3}{n}\right]$$

$$= 3[1+3+3]$$

$$= 3[7]$$

Hence, $I_1 = 21$ ...(ii) **[1 Mark]**

Again, $I_2 = \int_1^4 x dx$,

$a=1, b=4$ and $f(x)=x$

$$\Rightarrow h = \frac{4-1}{n} = \frac{3}{n}$$

$$\therefore I_2 = (4-1)\lim_{n\to\infty}\frac{1}{n}[f(1)+f(1+h)+\cdots f(a+(n-1)h)]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}[1+(1+h)+\cdots+(1+(n-1)h)]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[1+\left(1+\frac{3}{n}\right)+\cdots+\left\{1+(n-1)\frac{3}{n}\right\}\right]$$

$= 3\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{1+1+\cdots+1}_{n \text{ times}}\right)+\frac{3}{n}\left(1+2+\cdots+(n-1)\right)\right]$ **[1 Mark]**

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{3}{n}\left\{\frac{(n-1)n}{2}\right\}\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[1+\frac{3}{2}\left(1-\frac{1}{n}\right)\right]$$

$$= 3\left[1+\frac{3}{2}\right]$$

$$= 3\left[\frac{5}{2}\right]$$

Hence, $I_2 = \frac{15}{2}$ ...(iii) **[1 Mark]**

From equations (i), (ii) and (iii), we get

$I = I_1 - I_2 = 21 - \frac{15}{2} = \frac{27}{2}$ **[1 Mark]**

5. Evaluate the definite integral $\int_{-1}^{1} e^x dx$ as limit of sum. **[4 Marks]**

Solution:

Let $I = \int_{-1}^{1} e^x dx$ ...(i)

We know that,

$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a)+f(a+h)\ldots f(a+(n-1)h)]$, where $h = \frac{b-a}{n}$

Here we have, $a=-1, b=1$ and $f(x)=e^x$

$\therefore h = \frac{1+1}{n} = \frac{2}{n}$ **[1 Mark]**

$$\therefore I = (1+1)\lim_{n\to\infty}\frac{1}{n}\left[f(-1)+f\left(-1+\frac{2}{n}\right)+f\left(-1+2\cdot\frac{2}{n}\right)+\cdots+f\left(-1+\frac{(n-1)2}{n}\right)\right]$$

$= 2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}+e^{\left(-1+\frac{2}{n}\right)}+e^{\left(-1+2\frac{2}{n}\right)}+\cdots e^{\left(-1+(n-1)\frac{2}{n}\right)}\right]$ **[1 Mark]**

$= 2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}\left\{1 + e^{\frac{2}{n}} + e^{\frac{4}{n}} + e^{\frac{6}{n}} + e^{(n-1)\frac{2}{n}}\right\}\right]$

$= 2\lim_{n\to\infty}\frac{e^{-1}}{n}\left[\frac{e^{\frac{2n}{n}}-1}{e^{\frac{2}{n}}-1}\right]$

$= e^{-1} \times 2\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^{2}-1}{e^{\frac{2}{n}}-1}\right]$ **[1 Mark]**

$= \frac{e^{-1}\times 2(e^2-1)}{\lim_{\frac{2}{n}\to 0}\left(\frac{e^{\frac{2}{n}}-1}{\frac{2}{n}}\right)\times 2}$

$= e^{-1}\left[\frac{2(e^2-1)}{2}\right]$ $\left[\lim_{h\to 0}\left(\frac{e^h-1}{h}\right) = 1\right]$

$= \frac{e^2-1}{e}$

$= \left(e - \frac{1}{e}\right)$ **[1 Mark]**

6. Evaluate the definite integral $\int_0^4 (x + e^{2x})dx$ as limit of sum. **[6 Marks]**

Solution:

We know that,

$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a) + f(a+h) + \cdots + f(a+(n-1)h)]$, where $h = \frac{b-a}{n}$

Here we have, $a = 0, b = 4$ and $f(x) = x + e^{2x}$

$\therefore h = \frac{4-0}{n} = \frac{4}{n}$ **[1 Mark]**

$\Rightarrow \int_0^4 (x + e^{2x})\,dx = (4-0)\lim_{n\to\infty}\frac{1}{n}[f(0) + f(h) + f(2h) + \cdots + f((n-1)h)]$

$= 4\lim_{n\to\infty}\frac{1}{n}\left[(0 + e^0) + \left(h + e^{2h}\right) + \left(2h + e^{2\cdot 2h}\right) + \cdots + \left\{(n-1)h + e^{2(n-1)h}\right\}\right]$

$= 4\lim_{n\to\infty}\frac{1}{n}\left[1 + \left(h + e^{2h}\right) + \left(2h + e^{4h}\right) + \cdots + \left\{(n-1)h + e^{2(n-1)h}\right\}\right]$

$= 4\lim_{n\to\infty}\frac{1}{n}\left[\{h+2h+3h+\cdots+(n-1)h\}+\left(1+e^{2h}+e^{4h}+\cdots+e^{2(n-1)h}\right)\right]$ **[2 Marks]**

$= 4\lim_{n\to\infty}\frac{1}{n}\left[h\{1+2+\cdots(n-1)\}+\left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$

$= 4\lim_{n\to\infty}\frac{1}{n}\left[\frac{(h(n-1)n)}{2}+\left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$

$= 4\lim_{n\to\infty}\frac{1}{n}\left[\frac{4}{n}\cdot\frac{(n-1)n}{2}+\left(\frac{e^{8}-1}{e^{\frac{8}{n}}-1}\right)\right]$

$= 4(2)+4\lim_{n\to\infty}\frac{(e^{8}-1)}{\left(\frac{e^{\frac{8}{n}}-1}{\frac{8}{n}}\right)8}$ **[2 Marks]**

$= 8+\frac{4\cdot(e^{8}-1)}{8} \quad \left(\lim_{x\to 0}\frac{e^{x}-1}{x}=1\right)$

$= 8+\frac{e^{8}-1}{2}$

$= \frac{15+e^{8}}{2}$ **[1 Mark]**

Exercise 7.9

1. Evaluate the definite integral $\int_{-1}^{1}(x+1)dx$ **[2 Marks]**

Solution:

Let $I=\int_{-1}^{1}(x+1)dx$

$\int(x+1)dx=\frac{x^{2}}{2}+x=F(x)$ **[1 Mark]**

By second fundamental theorem of calculus, we get

$I=F(1)-F(-1)$

$=\left(\frac{1}{2}+1\right)-\left(\frac{1}{2}-1\right)$

$=\frac{1}{2}+1-\frac{1}{2}+1$

$= 2$ [1 Mark]

2. Evaluate the definite integral $\int_2^3 \frac{1}{x} dx$ [2 Marks]

Solution:

Let $I = \int_2^3 \frac{1}{x} dx$

$\int \frac{1}{x} dx = \log|x| = F(x)$ [1 Mark]

By second fundamental theorem of calculus, we get

$I = F(3) - F(2)$

$= \log|3| - \log|2| = \log\frac{3}{2}$ [1 Mark]

3. Evaluate the definite integral $\int_1^2 (4x^3 - 5x^2 + 6x + 9)dx$ [2 Marks]

Solution:

Let $I = \int_1^2 (4x^3 - 5x^2 + 6x + 9)\, dx$

$\int (4x^3 - 5x^2 + 6x + 9)dx = 4\left(\frac{x^4}{4}\right) - 5\left(\frac{x^3}{3}\right) + 6\left(\frac{x^2}{2}\right) + 9(x)$

$= x^4 - \frac{5x^3}{3} + 3x^2 + 9x = F(x)$ [1 Mark]

By second fundamental theorem of calculus, we get

$I = F(2) - F(1)$

$I = \left\{2^4 - \frac{5\cdot(2)^3}{3} + 3(2)^2 + 9(2)\right\} - \left\{(1)^4 - \frac{5(1)^3}{3} + 3(1)^2 + 9(1)\right\}$

$= \left(16 - \frac{40}{3} + 12 + 18\right) - \left(1 - \frac{5}{3} + 3 + 9\right)$

$= 16 - \frac{40}{3} + 12 + 18 - 1 + \frac{5}{3} - 3 - 9$

$= 33 - \frac{35}{3}$

$= \frac{99-35}{3}$

$= \frac{64}{3}$ **[1 Mark]**

4. Evaluate the definite integral $\int_0^{\frac{\pi}{4}} \sin 2x\, dx$ **[2 Marks]**

Solution:

Let $I = \int_0^{\frac{\pi}{4}} \sin 2x\, dx$

$\int \sin 2x\, dx = \left(\frac{-\cos 2x}{2}\right) = F(x)$ **[1 Mark]**

By second fundamental theorem of calculus, we get

$I = F\left(\frac{\pi}{4}\right) - F(0)$

$= -\frac{1}{2}\left[\cos 2\left(\frac{\pi}{4}\right) - \cos 0\right]$

$= -\frac{1}{2}\left[\cos\left(\frac{\pi}{2}\right) - \cos 0\right]$

$= -\frac{1}{2}[0 - 1]$

$= \frac{1}{2}$ **[1 Mark]**

5. Evaluate the definite integral $\int_0^{\frac{\pi}{2}} \cos 2x\, dx$ **[2 Marks]**

Solution:

Let $I = \int_0^{\frac{\pi}{2}} \cos 2x \, dx$

$\int \cos 2x \, dx = \left(\frac{\sin 2x}{2}\right) = F(x)$ **[1 Mark]**

By second fundamental theorem of calculus, we get

$I = \text{F}\left(\frac{\pi}{2}\right) - \text{F}(0)$

$= \frac{1}{2}\left[\sin 2\left(\frac{\pi}{2}\right) - \sin 0\right]$

$= \frac{1}{2}[\sin \pi - \sin 0]$

$= \frac{1}{2}[0 - 0] = 0$ **[1 Mark]**

6. Evaluate the definite integral $\int_4^5 e^x dx$ **[2 Marks]**

Solution:

Let $I = \int_4^5 e^x dx$

$\int e^x dx = e^x = F(x)$ **[1 Mark]**

By second fundamental theorem of calculus, we get

$I = F(5) - F(4)$

$= e^5 - e^4$

$= e^4(e - 1)$ **[1 Mark]**

7. Evaluate the definite integral $\int_0^{\frac{\pi}{4}} \tan x \, dx$ **[2 Marks]**

Solution:

Let $I = \int_0^{\frac{\pi}{4}} \tan x \, dx$

$\int \tan x \, dx = -\log|\cos x| = F(x)$ **[1 Mark]**

By second fundamental theorem of calculus, we get

$I = F\left(\frac{\pi}{4}\right) - F(0)$

$= -\log\left|\cos\frac{\pi}{4}\right| + \log|\cos 0|$

$= -\log\left|\frac{1}{\sqrt{2}}\right| + \log|1|$

$= -\log(2)^{-\frac{1}{2}}$

$= \frac{1}{2}\log 2$ **[1 Mark]**

8. Evaluate the definite integral $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \, dx$ **[2 Marks]**

Solution:

Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \, dx$

$\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| = F(x)$ **[1 Mark]**

By second fundamental theorem of calculus, we get

$I = \mathrm{F}\left(\frac{\pi}{4}\right) - \mathrm{F}\left(\frac{\pi}{6}\right)$

$= \log\left|\operatorname{cosec}\frac{\pi}{4} - \cot\frac{\pi}{4}\right| - \log\left|\operatorname{cosec}\frac{\pi}{6} - \cot\frac{\pi}{6}\right|$

$= \log\left|\sqrt{2} - 1\right| - \log\left|2 - \sqrt{3}\right|$

$= \log\left(\frac{\sqrt{2}-1}{2-\sqrt{3}}\right)$ **[1 Mark]**

9. Evaluate the definite integral $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ **[2 Marks]**

Solution:

Let $I = \int_0^1 \frac{dx}{\sqrt{1-x^2}}$

$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x = F(x)$ **[1 Mark]**

By second fundamental theorem of calculus, we get

$I = F(1) - F(0)$

$= \sin^{-1}(1) - \sin^{-1}(0)$

$= \frac{\pi}{2} - 0$

$= \frac{\pi}{2}$ **[1 Mark]**

10. Evaluate the definite integral $\int_0^1 \frac{dx}{1+x^2}$ **[2 Marks]**

Solution:

Let $I = \int_0^1 \frac{dx}{1+x^2}$

$\int \frac{dx}{1+x^2} = \tan^{-1} x = F(x)$ **[1 Mark]**

By second fundamental theorem of calculus, we get

$I = F(1) - F(0)$

$= \tan^{-1}(1) - \tan^{-1}(0)$

$= \frac{\pi}{4}$ **[1 Mark]**

11. Evaluate the definite integral $\int_2^3 \frac{dx}{x^2-1}$ **[2 Marks]**

Solution:

Let $I = \int_2^3 \frac{dx}{x^2-1}$

$\int \frac{dx}{x^2-1} = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| = F(x)$ **[1 Mark]**

By second fundamental theorem of calculus, we get

$I = F(3) - F(2)$

$= \frac{1}{2}\left[\log\left|\frac{3-1}{3+1}\right| - \log\left|\frac{2-1}{2+1}\right|\right]$

$= \frac{1}{2}\left[\log\left|\frac{2}{4}\right| - \log\left|\frac{1}{3}\right|\right]$

$= \frac{1}{2}\left[\log\frac{1}{2} - \log\frac{1}{3}\right]$

$= \frac{1}{2}\left[\log\frac{3}{2}\right]$ **[1 Mark]**

12. Evaluate the definite integral $\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$ **[2 Marks]**

Solution:

Let $I = \int_0^{\frac{\pi}{2}} \cos^2 x\, dx$

$\int \cos^2 x\, dx = \int \left(\frac{1+\cos 2x}{2}\right) dx = \frac{x}{2} + \frac{\sin 2x}{4} = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) = F(x)$ **[1 Mark]**

By second fundamental theorem of calculus, we get

$I = \left[F\left(\frac{\pi}{2}\right) - F(0)\right]$

$= \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{\sin\pi}{2}\right) - \left(0 + \frac{\sin 0}{2}\right)\right]$

$= \frac{1}{2}\left[\frac{\pi}{2} + 0 - 0 - 0\right]$

$= \frac{\pi}{4}$ **[1 Mark]**

13. Evaluate the definite integral $\int_2^3 \frac{xdx}{x^2+1}$ **[2 Marks]**

Solution:

Let $I = \int_2^3 \frac{x}{x^2+1} dx$

$\int \frac{x}{x^2+1} dx = \frac{1}{2}\int \frac{2x}{x^2+1} dx = \frac{1}{2}\log(1+x^2) = F(x)$ **[1 Mark]**

By second fundamental theorem of calculus, we get

$I = F(3) - F(2)$

$= \frac{1}{2}[\log(1+(3)^2) - \log(1+(2)^2)]$

$= \frac{1}{2}[\log(10) - \log(5)]$

$= \frac{1}{2}\log\left(\frac{10}{5}\right) = \frac{1}{2}\log 2$ **[1 Mark]**

14. Evaluate the definite integral $\int_0^1 \frac{2x+3}{5x^2+1} dx$ **[4 Marks]**

Solution:

Let $I = \int_0^1 \frac{2x+3}{5x^2+1} dx$

$$\int \frac{2x+3}{5x^2+1} dx = \frac{1}{5}\int \frac{5(2x+3)}{5x^2+1} dx$$

$= \frac{1}{5}\int \frac{10x+15}{5x^2+1} dx$

$= \frac{1}{5}\int \frac{10x}{5x^2+1} dx + 3\int \frac{1}{5x^2+1} dx$

$= \frac{1}{5}\int \frac{10x}{5x^2+1} dx + 3\int \frac{1}{5\left(x^2+\frac{1}{5}\right)} dx$ **[2 Marks]**

$= \frac{1}{5}\log(5x^2+1) + \frac{3}{5}\cdot\frac{1}{\frac{1}{\sqrt{5}}}\tan^{-1}\frac{x}{\frac{1}{\sqrt{5}}}$

$= \frac{1}{5}\log(5x^2+1) + \frac{3}{\sqrt{5}}\tan^{-1}(\sqrt{5}x)$

$= F(x)$ **[1 Mark]**

By second fundamental theorem of calculus, we get

$I = F(1) - F(0)$

$= \left\{\frac{1}{5}\log(5+1) + \frac{3}{\sqrt{5}}\tan^{-1}(\sqrt{5})\right\} - \left\{\frac{1}{5}\log(1) + \frac{3}{\sqrt{5}}\tan^{-1}(0)\right\}$

$= \frac{1}{5}\log 6 + \frac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}$ **[1 Mark]**

15. Evaluate the definite integral $\int_0^1 xe^{x^2}dx$ **[2 Marks]**

Solution:

Let $I = \int_0^1 xe^{x^2}dx$

Substitute $x^2 = t \Rightarrow 2x\,dx = dt$

When $x = 0, t = 0$ and when $x = 1, t = 1,$

$\therefore I = \frac{1}{2}\int_0^1 e^t dt$

$\frac{1}{2}\int e^t dt = \frac{1}{2}e^t = F(t)$ **[1 Mark]**

By second fundamental theorem of calculus, we get

$I = F(1) - F(0)$

$= \frac{1}{2}e - \frac{1}{2}e^0$

$= \frac{1}{2}(e-1)$ **[1 Mark]**

16. Evaluate the definite integral $\int_1^2 \frac{5x^2}{x^2+4x+3}dx$ **[4 Marks]**

Solution:

Let $I = \int_1^2 \frac{5x^2}{x^2+4x+3}dx$

Dividing $5x^2$ by $x^2 + 4x + 3$, we get

$I = \int_1^2 \left\{5 - \frac{20x+15}{x^2+4x+3}\right\}dx$

$= \int_1^2 5dx - \int_1^2 \frac{20x+15}{x^2+4x+3}dx$

$= [5x]_1^2 - \int_1^2 \frac{20x+15}{x^2+4x+3}dx$

$= 5 \times 2 - 5 - \int_1^2 \frac{20x+15}{x^2+4x+3}dx$

$= 5 - \int_1^2 \frac{20x+15}{x^2+4x+3}dx$ **[1 Mark]**

Let $I = 5 - I_1$, where $I_1 = \int_1^2 \frac{20x+15}{x^2+4x+3}dx$ ...(i)

Now, consider $I_1 = \int_1^2 \frac{20x+15}{x^2+4x+8}dx$

Let $20x + 15 = A\frac{d}{dx}(x^2 + 4x + 3) + B$

$= 2Ax + (4A + B)$

Equating the coefficients of x and constant term, we get

$A = 10$ and $B = -25$

$\Rightarrow I_1 = 10\int_1^2 \frac{2x+4}{x^2+4x+3}dx - 25\int_1^2 \frac{dx}{x^2+4x+3}$

Let $x^2 + 4x + 3 = t$

$\Rightarrow (2x + 4)dx = dt$ **[1 Mark]**

$$\Rightarrow I_1 = 10\int \frac{dt}{t} - 25\int \frac{dx}{(x+2)^2 - 1^2}$$

$$= 10\log t - 25\left[\frac{1}{2}\log\left(\frac{x+2-1}{x+2+1}\right)\right]$$

$$= [10\log(x^2 + 4x + 3)]_1^2 - 25\left[\frac{1}{2}\log\left(\frac{x+1}{x+3}\right)\right]_1^2$$

$= [10\log 15 - 10\log 8] - 25\left[\frac{1}{2}\log\frac{3}{5} - \frac{1}{2}\log\frac{2}{4}\right]$

$= [10\log(5\times 3) - 10\log(4\times 2)] - \frac{25}{2}[\log 3 - \log 5 - \log 2 + \log 4]$

$= [10\log 5 + 10\log 3 - 10\log 4 - 10\log 2] - \frac{25}{2}[\log 3 - \log 5 - \log 2 + \log 4]$

$= \left[10 + \frac{25}{2}\right]\log 5 + \left[-10 - \frac{25}{2}\right]\log 4 + \left[10 - \frac{25}{2}\right]\log 3 + \left[-10 + \frac{25}{2}\right]\log 2$

$= \frac{45}{2}\log 5 - \frac{45}{2}\log 4 - \frac{5}{2}\log 3 + \frac{5}{2}\log 2$

$= \frac{45}{2}\log\frac{5}{4} - \frac{5}{2}\log\frac{3}{2}$ **[1 Mark]**

Substituting the value of I_1 in (i), we get

$I = 5 - \left[\frac{45}{2}\log\frac{5}{4} - \frac{5}{2}\log\frac{3}{2}\right]$

$= 5 - \frac{5}{2}\left[9\log\frac{5}{4} - \log\frac{3}{2}\right]$ **[1 Mark]**

17. Evaluate the definite integral $\int_0^{\frac{\pi}{4}}(2\sec^2 x + x^3 + 2)\,dx$ **[2 Marks]**

Solution:

Let $I = \int_0^{\frac{\pi}{4}}(2\sec^2 x + x^3 + 2)\,dx$

$\int(2\sec^2 x + x^3 + 2)dx = 2\tan x + \frac{x^4}{4} + 2x = F(x)$ **[1 Mark]**

By second fundamental theorem of calculus, we get

$I = F\left(\frac{\pi}{4}\right) - F(0)$

$= \left\{\left(2\tan\frac{\pi}{4} + \frac{1}{4}\left(\frac{\pi}{4}\right)^4 + 2\left(\frac{\pi}{4}\right)\right) - (2\tan 0 + 0 + 0)\right\}$

$= 2\tan\frac{\pi}{4} + \frac{\pi^4}{4^5} + \frac{\pi}{2}$

$= 2 + \frac{\pi}{2} + \frac{\pi^4}{1024}$ **[1 Mark]**

18. Evaluate the definite integral $\int_0^{\pi}\left(\sin^2\frac{x}{2}-\cos^2\frac{x}{2}\right)dx$ **[2 Marks]**

Solution:

Let $I=\int_0^{\pi}\left(\sin^2\frac{x}{2}-\cos^2\frac{x}{2}\right)dx$

$$=-\int_0^{\pi}\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)dx$$

$$=-\int_0^{\pi}\cos x\,dx$$

$-\int\cos x\,dx=-\sin x=F(x)$ **[1 Mark]**

By second fundamental theorem of calculus, we get

$I=F(\pi)-F(0)$

$=-\sin\pi+\sin 0$

$=0$ **[1 Mark]**

19. Evaluate the definite integral $\int_0^{2}\frac{6x+3}{x^2+4}dx$ **[4 Marks]**

Solution:

Let $I=\int_0^{2}\frac{6x+3}{x^2+4}dx$

$\int\frac{6x+3}{x^2+4}dx=3\int\frac{2x+1}{x^2+4}dx$

$=3\int\frac{2x}{x^2+4}dx+3\int\frac{1}{x^2+4}dx$ **[1 Mark]**

$=3\log(x^2+4)+\frac{3}{2}\tan^{-1}\frac{x}{2}=F(x)$ **[1 Mark]**

By second fundamental theorem of calculus, we get

$I = F(2) - F(0)$

$= \left\{3\log(2^2+4) + \frac{3}{2}\tan^{-1}\left(\frac{2}{2}\right)\right\} - \left\{3\log(0+4) + \frac{3}{2}\tan^{-1}\left(\frac{0}{2}\right)\right\}$

$= 3\log 8 + \frac{3}{2}\tan^{-1} 1 - 3\log 4 - \frac{3}{2}\tan^{-1} 0$

$= 3\log 8 + \frac{3}{2}\left(\frac{\pi}{4}\right) - 3\log 4 - 0$

$= 3\log\left(\frac{8}{4}\right) + \frac{3\pi}{8}$

$= 3\log 2 + \frac{3\pi}{8}$ **[2 Marks]**

20. Evaluate the definite integral $\int_0^1 \left(xe^x + \sin\frac{\pi x}{4}\right) dx$ **[2 Marks]**

Solution:

Let $I = \int_0^1 \left(xe^x + \sin\frac{\pi x}{4}\right) dx$

$\int \left(xe^x + \sin\frac{\pi x}{4}\right) dx = x\int e^x dx - \int\left\{\left(\frac{d}{dx}x\right)\int e^x dx\right\} dx + \left\{\frac{-\cos\frac{\pi x}{4}}{\frac{\pi}{4}}\right\}$

$= xe^x - \int e^x dx - \frac{4}{\pi}\cos\frac{\pi x}{4}$

$= xe^x - e^x - \frac{4}{\pi}\cos\frac{\pi x}{4}$

$= F(x)$ **[1 Mark]**

By second fundamental theorem of calculus, we get

$I = F(1) - F(0)$

$= \left(1.e^1 - e^1 - \frac{4}{\pi}\cos\frac{\pi}{4}\right) - \left(0.e^0 - e^0 - \frac{4}{\pi}\cos 0\right)$

$= e - e - \frac{4}{\pi}\left(\frac{1}{\sqrt{2}}\right) + 1 + \frac{4}{\pi}$

$= 1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}$ **[1 Mark]**

21. $\int_{1}^{\sqrt{3}} \frac{dx}{1+x^2}$ equals **[2 Marks]**

(A) $\frac{\pi}{3}$

(B) $\frac{2\pi}{3}$

(C) $\frac{\pi}{6}$

(D) $\frac{\pi}{12}$

Solution:

$\int \frac{dx}{1+x^2} = \tan^{-1} x = F(x)$ **[1 Mark]**

By second fundamental theorem of calculus, we get

$\int_{1}^{\sqrt{3}} \frac{dx}{1+x^2} = F(\sqrt{3}) - F(1)$

$= \tan^{-1} \sqrt{3} - \tan^{-1} 1$

$= \frac{\pi}{3} - \frac{\pi}{4}$

$= \frac{\pi}{12}$

Hence, (D) is the correct answer. **[1 Mark]**

22. $\int_{0}^{\frac{2}{3}} \frac{dx}{4+9x^2}$ equals **[2 Marks]**

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{12}$

(C) $\frac{\pi}{24}$

(D) $\frac{\pi}{4}$

Solution:

$\int \frac{dx}{4+9x^2} = \int \frac{dx}{(2)^2+(3x)^2}$

Put $3x = t \Rightarrow 3dx = dt$

$\therefore \int \frac{dx}{(2)^2+(3x)^2} = \frac{1}{3}\int \frac{dt}{(2)^2+t^2}$

$= \frac{1}{3}\left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]$

$= \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right)$

$= F(x)$ **[1 Mark]**

By second fundamental theorem of calculus, we get

$\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2} = F\left(\frac{2}{3}\right) - F(0)$

$= \frac{1}{6}\tan^{-1}\left(\frac{3}{2}\cdot\frac{2}{3}\right) - \frac{1}{6}\tan^{-1} 0$

$= \frac{1}{6}\tan^{-1} 1 - 0$

$= \frac{1}{6} \times \frac{\pi}{4}$

$= \frac{\pi}{24}$

Hence, (C) is the correct answer. **[1 Mark]**

Exercise 7.10

1. Evaluate the integral $\int_0^1 \frac{x}{x^2+1} dx$ using substitution. **[2 Marks]**

Solution:

$$\int_0^1 \frac{x}{x^2+1} dx$$

Let $x^2 + 1 = t \Rightarrow 2x\, dx = dt$

When $x = 0, t = 1$, when $x = 1, t = 2$

$\therefore \int_0^1 \frac{x}{x^2+1} dx = \frac{1}{2}\int_1^2 \frac{dt}{t}$ [1 Mark]

$= \frac{1}{2}[\log|t|]_1^2$

$= \frac{1}{2}[\log 2 - \log 1]$

$= \frac{1}{2}\log 2$ [1 Mark]

2. Evaluate the integral $\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi d\phi$ using substitution. [2 Marks]

Solution:

Let $I = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi d\phi = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^4\phi\cos\phi d\phi$

Again, let $\sin\phi = t \Rightarrow \cos\phi d\phi = dt$

When $\phi = 0, t = 0$, when $\phi = \frac{\pi}{2}, t = 1$

$\therefore I = \int_0^1 \sqrt{t}(1-t^2)^2 dt$ [1 Mark]

$= \int_0^1 t^{\frac{1}{2}}(1+t^4-2t^2)dt$

$= \int_0^1 \left[t^{\frac{1}{2}} + t^{\frac{9}{2}} - 2t^{\frac{5}{2}}\right]dt$

$= \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{t^{\frac{11}{2}}}{\frac{11}{2}} - \frac{2t^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^1$

$= \frac{2}{3} + \frac{2}{11} - \frac{4}{7} - 0$

$= \frac{154+42-132}{231}$

$= \frac{64}{231}$ [1 Mark]

3. Evaluate the integral $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$ using substitution. **[4 Marks]**

Solution:

Let $I = \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Again, let $x = \tan\theta \Rightarrow dx = \sec^2\theta d\theta$

When $x = 0, \theta = 0$, when $x = 1, \theta = \frac{\pi}{4}$

$I = \int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta d\theta$ **[1 Mark]**

$= \int_0^{\frac{\pi}{4}} \sin^{-1}(\sin 2\theta)\sec^2\theta d\theta$

$= \int_0^{\frac{\pi}{4}} 2\theta \sec^2\theta d\theta$

$= 2\int_0^{\frac{\pi}{4}} \theta \sec^2\theta d\theta$ **[1 Mark]**

Taking θ as first function and $\sec^2\theta$ as second function and integrating using by parts, we get

$I = 2\left[\theta\int\sec^2\theta d\theta - \int\left\{\left(\frac{d}{dx}\theta\right)\int\sec^2\theta d\theta\right\}d\theta\right]_0^{\frac{\pi}{4}}$ **[1 Mark]**

$= 2[\theta\tan\theta - \int\tan\theta d\theta]_0^{\frac{\pi}{4}}$

$= 2[\theta\tan\theta + \log|\cos\theta|]_0^{\frac{\pi}{4}}$

$= 2\left[\frac{\pi}{4}\tan\frac{\pi}{4} + \log\left|\cos\frac{\pi}{4}\right| - \log|\cos 0|\right]$

$= 2\left[\frac{\pi}{4} + \log\left(\frac{1}{\sqrt{2}}\right) - \log 1\right]$

$= 2\left[\frac{\pi}{4} - \frac{1}{2}\log 2\right]$

$= \frac{\pi}{2} - \log 2$ **[1 Mark]**

4. Evaluate the integral $\int_0^2 x\sqrt{x+2}$ using substitution. (Put $x+2=t^2$) **[2 Marks]**

Solution:

$\int_0^2 x\sqrt{x+2}\,dx$

Let $x+2=t^2 \Rightarrow dx=2tdt$

When $x=0, t=\sqrt{2}$, when $x=2, t=2$

$\therefore \int_0^2 x\sqrt{x+2}\,dx=\int_{\sqrt{2}}^2 (t^2-2)\sqrt{t^2}\,2tdt$ **[1 Mark]**

$=2\int_{\sqrt{2}}^2 (t^4-2t^2)\,dt$

$=2\left[\frac{t^5}{5}-\frac{2t^3}{3}\right]_{\sqrt{2}}^2$

$=2\left[\frac{32}{5}-\frac{16}{3}-\frac{4\sqrt{2}}{5}+\frac{4\sqrt{2}}{3}\right]$

$=2\left[\frac{96-80-12\sqrt{2}+20\sqrt{2}}{15}\right]$

$=2\left[\frac{16+8\sqrt{2}}{15}\right]$

$=\frac{16(2+\sqrt{2})}{15}$

$=\frac{16\sqrt{2}(\sqrt{2}+1)}{15}$ **[1 Mark]**

5. Evaluate the integral $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx$ using substitution. **[2 Marks]**

Solution:

$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx$

Let $\cos x=t \Rightarrow -\sin x\,dx=dt$

When $x=0, t=1$, when $x=\frac{\pi}{2}, t=0$

$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx = -\int_1^0 \frac{dt}{1+t^2}$ **[1 Mark]**

$= -[\tan^{-1} t]_1^0$

$= -[\tan^{-1} 0 - \tan^{-1} 1]$

$= -\left[-\frac{\pi}{4}\right]$

$= \frac{\pi}{4}$ **[1 Mark]**

6. Evaluate the integral $\int_0^2 \frac{dx}{x+4-x^2}$ using substitution. **[4 Marks]**

Solution:

$\int_0^2 \frac{dx}{x+4-x^2} = \int_0^2 \frac{dx}{-(x^2-x-4)}$

$= \int_0^2 \frac{dx}{-\left(x^2-x+\frac{1}{4}-\frac{1}{4}-4\right)}$

$= \int_0^2 \frac{dx}{-\left[\left(x-\frac{1}{2}\right)^2-\frac{17}{4}\right]}$

$= \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}$ **[1 Mark]**

Let $x - \frac{1}{2} = t \quad \Rightarrow dx = dt$

When $x = 0, t = -\frac{1}{2}$, when $x = 2, t = \frac{3}{2}$

$\therefore \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2} = \int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{dt}{\left(\frac{\sqrt{17}}{2}\right)^2-t^2}$ **[1 Mark]**

$= \left[\frac{1}{2\left(\frac{\sqrt{17}}{2}\right)} \log \frac{\frac{\sqrt{17}}{2}+t}{\frac{\sqrt{17}}{2}-t}\right]_{-\frac{1}{2}}^{\frac{3}{2}}$

$= \frac{1}{\sqrt{17}}\left[\log \frac{\frac{\sqrt{17}}{2}+\frac{3}{2}}{\frac{\sqrt{17}}{2}-\frac{3}{2}} - \log \frac{\frac{\sqrt{17}}{2}-\frac{1}{2}}{\frac{\sqrt{17}}{2}+\frac{1}{2}}\right]$

$= \frac{1}{\sqrt{17}}\left[\log\frac{\sqrt{17}+3}{\sqrt{17}-3} - \log\frac{\sqrt{17}-1}{\sqrt{17}+1}\right]$

$= \frac{1}{\sqrt{17}}\log\frac{\sqrt{17}+3}{\sqrt{17}-3} \times \frac{\sqrt{17}+1}{\sqrt{17}-1}$

$= \frac{1}{\sqrt{17}}\log\left[\frac{17+3+4\sqrt{17}}{17+3-4\sqrt{17}}\right]$

$= \frac{1}{\sqrt{17}}\log\left[\frac{20+4\sqrt{17}}{20-4\sqrt{17}}\right]$

$= \frac{1}{\sqrt{17}}\log\left(\frac{5+\sqrt{17}}{5-\sqrt{17}}\right)$

$= \frac{1}{\sqrt{17}}\log\left[\frac{(5+\sqrt{17})(5+\sqrt{17})}{25-17}\right]$

$= \frac{1}{\sqrt{17}}\log\left[\frac{25+17+10\sqrt{17}}{8}\right]$

$= \frac{1}{\sqrt{17}}\log\left(\frac{42+10\sqrt{17}}{8}\right)$

$= \frac{1}{\sqrt{17}}\log\left(\frac{21+5\sqrt{17}}{4}\right)$ **[2 Marks]**

7. Evaluate the integral $\int_{-1}^{1}\frac{dx}{x^2+2x+5}$ using substitution. **[4 Marks]**

Solution:

$\int_{-1}^{1}\frac{dx}{x^2+2x+5} = \int_{-1}^{1}\frac{dx}{(x^2+2x+1)+4} = \int_{-1}^{1}\frac{dx}{(x+1)^2+(2)^2}$ **[1 Mark]**

Let $x+1 = t \Rightarrow dx = dt$

When $x = -1, t = 0$, when $x = 1, t = 2$

$\therefore \int_{-1}^{1}\frac{dx}{(x+1)^2+(2)^2} = \int_{0}^{2}\frac{dt}{t^2+2^2}$ **[1 Mark]**

$= \left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]_0^2$

$= \frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}0$

$= \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}$ **[2 Marks]**

8. Evaluate the integral $\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx$ using substitution. **[4 Marks]**

Solution:

$\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx$

Let $2x = t \Rightarrow 2dx = dt$

When $x = 1, t = 2$, when $x = 2, t = 4$

$\therefore \int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx = \frac{1}{2}\int_2^4 \left(\frac{2}{t} - \frac{2}{t^2}\right) e^t dt$

$= \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2}\right) e^t dt$ **[2 Marks]**

Let $\frac{1}{t} = f(t)$

$\Rightarrow f'(t) = -\frac{1}{t^2}$

$\Rightarrow \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2}\right) e^t dt = \int_2^4 e^t [f(t) + f'(t)] dt$

$= [e^t f(t)]_2^4$

$= \left[e^t \cdot \frac{1}{t}\right]_2^4$

$= \left[\frac{e^t}{t}\right]_2^4$

$= \frac{e^4}{4} - \frac{e^2}{2}$

$= \frac{e^2(e^2-2)}{4}$ **[2 Marks]**

9. The value of the integral $\int_{\frac{1}{3}}^{1} \frac{(x-x^3)^{\frac{1}{3}}}{x^4} dx$ is **[4 Marks]**

(A) 6

(B) 0

(C) 3

(D) 4

Solution:

Let $I = \int_{\frac{1}{3}}^{1} \frac{(x-x^3)^{\frac{1}{3}}}{x^4} dx = \int_{\frac{1}{3}}^{1} \frac{x\left(\frac{1}{x^2}-1\right)^{\frac{1}{3}}}{x^4} dx = \int_{\frac{1}{3}}^{1} \frac{\left(\frac{1}{x^2}-1\right)^{\frac{1}{3}}}{x^3} dx$ **[1 Mark]**

Again, let $\frac{1}{x^2} - 1 = t \ \Rightarrow \frac{-2}{x^3} dx = dt$

$\Rightarrow \frac{dx}{x^3} = -\frac{1}{2} dt$ **[1 Mark]**

When $x = \frac{1}{3}, t = 8$, when $x = 1, t = 0$

$\Rightarrow I = -\frac{1}{2} \int_8^0 t^{\frac{1}{3}} dt$ **[1 Mark]**

$= -\frac{1}{2} \left[\frac{t^{\frac{4}{3}}}{\frac{4}{3}} \right]_8^0$

$= -\frac{1}{2} [0 - \frac{3}{4} \times 16]$

$= 6$

Hence, (A) is the correct answer. **[1 Mark]**

10. If $f(x) = \int_0^x t \sin t \, dt$, then $f'(x)$ is **[4 Marks]**

(A) $\cos x + x \sin x$

(B) $x \sin x$

(C) $x \cos x$

(D) $\sin x + x \cos x$

Solution:

$$f(x)=\int_0^x t\sin t dt$$

Using integration by parts, we get

$f(x)=t\int_0^x \sin t\,dt-\int_0^x\left\{\left(\frac{d}{dt}t\right)\int \sin t\,dt\right\}dt$ **[1 Mark]**

$=[t(-\cos t)]_0^x-\int_0^x(-\cos t)\,dt$

$=[-t\cos t+\sin t]_0^x$ **[1 Mark]**

$=-x\cos x+\sin x$

$\Rightarrow f'(x)=-[\{x(-\sin x)\}+\cos x]+\cos x$ **[1 Mark]**

$=x\sin x-\cos x+\cos x$

$=x\sin x$

Hence, (B) is the correct answer. **[1 Mark]**

Exercise 7.11

1. By using the properties of definite integrals, evaluate the integral $\int_0^{\frac{\pi}{2}}\cos^2 x\,dx$ **[2 Marks]**

Solution:

Let $I=\int_0^{\frac{\pi}{2}}\cos^2 x\,dx$ ...(i)

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\cos^2\left(\frac{\pi}{2}-x\right)dx\quad\left(\because\int_0^a f(x)dx=\int_0^a f(a-x)\,dx\right)$$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\sin^2 x\,dx$ ...(ii) **[1 Mark]**

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} (\sin^2 x + \cos^2 x) dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1\, dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$\Rightarrow I = \frac{\pi}{4}$ **[1 Mark]**

2. By using the properties of definite integrals, evaluate the integral $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$ **[2 Marks]**

Solution:

Let $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$ ...(i)

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} dx \quad \left(\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx$ ...(ii) **[1 Mark]**

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1\, dx$$

$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$

$\Rightarrow 2I = \frac{\pi}{2}$

$\Rightarrow I = \frac{\pi}{4}$ **[1 Mark]**

3. By using the properties of definite integrals, evaluate the integral $\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x dx}{\sin^{\frac{3}{2}} x+\cos^{\frac{3}{2}} x}$ **[2 Marks]**

Solution:

Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x+\cos^{\frac{3}{2}} x} dx$ ...(i)

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)}{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)+\cos^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)} dx \quad \left(\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x+\cos^{\frac{3}{2}} x} dx$ ...(ii) **[1 Mark]**

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x+\cos^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x+\cos^{\frac{3}{2}} x} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 \cdot dx$$

$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$

$\Rightarrow 2I = \frac{\pi}{2}$

$\Rightarrow I = \frac{\pi}{4}$ **[1 Mark]**

4. By using the properties of definite integrals, evaluate the integral $\int_0^{\frac{\pi}{2}} \frac{\cos^5 x dx}{\sin^5 x+\cos^5 x}$ **[2 Marks]**

Solution:

Let $I = \int_0^{\frac{\pi}{2}} \frac{\cos^5 x}{\sin^5 x+\cos^5 x} dx$ ...(i)

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^5\left(\frac{\pi}{2}-x\right)}{\sin^5\left(\frac{\pi}{2}-x\right)+\cos^5\left(\frac{\pi}{2}-x\right)} dx \quad \left(\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x}{\sin^5 x+\cos^5 x} dx$ ...(ii) **[1 Mark]**

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x+\cos^5 x}{\sin^5 x+\cos^5 x} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 \cdot dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$\Rightarrow I = \frac{\pi}{4}$ **[1 Mark]**

5. By using the properties of definite integrals, evaluate the integral $\int_{-5}^{5} |x+2| dx$ **[4 Marks]**

Solution:

Let $I = \int_{-5}^{5} |x+2| dx$

It is seen that $(x+2) \le 0$ on $[-5,-2]$ and $(x+2) \ge 0$ on $[-2,5]$.

$\therefore I = \int_{-5}^{-2} -(x+2)dx + \int_{-2}^{5}(x+2)dx \quad \left(\because \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx\right)$ **[1 Mark]**

$I = -\left[\frac{x^2}{2} + 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{5}$ **[1 Mark]**

$= -\left[\frac{(-2)^2}{2} + 2(-2) - \frac{(-5)^2}{2} - 2(-5)\right] + \left[\frac{(5)^2}{2} + 2(5) - \frac{(-2)^2}{2} - 2(-2)\right]$

$= -\left[2 - 4 - \frac{25}{2} + 10\right] + \left[\frac{25}{2} + 10 - 2 + 4\right]$

$= -2 + 4 + \frac{25}{2} - 10 + \frac{25}{2} + 10 - 2 + 4$

$= 29$ **[2 Marks]**

6. By using the properties of definite integrals, evaluate the integral $\int_2^8 |x-5|dx$ **[4 Marks]**

Solution:

Let $I = \int_2^8 |x-5|dx$

It is seen that $(x-5) \leq 0$ on $[2,5]$ and $(x-5) \geq 0$ on $[5,8]$. **[1 Mark]**

Hence, $I = \int_2^5 -(x-5)dx + \int_2^8 (x-5)dx \quad \left(\because \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx\right)$ **[1 Mark]**

$= -\left[\frac{x^2}{2} - 5x\right]_2^5 + \left[\frac{x^2}{2} - 5x\right]_5^8$

$= -\left[\frac{25}{2} - 25 - 2 + 10\right] + \left[32 - 40 - \frac{25}{2} + 25\right]$

$= 9$ **[2 Marks]**

7. By using the properties of definite integrals, evaluate the integral $\int_0^1 x(1-x)^n\, dx$ **[2 Marks]**

Solution:

Let $I = \int_0^1 x(1-x)^n \, dx$

$\therefore I = \int_0^1 (1-x)\big(1-(1-x)\big)^n \, dx \qquad \left(\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$ **[1 Mark]**

$= \int_0^1 (1-x)(x)^n \, dx$

$= \int_0^1 (x^n - x^{n+1}) \, dx$

$= \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1$

$= \left[\frac{1}{n+1} - \frac{1}{n+2}\right]$

$= \frac{(n+2)-(n+1)}{(n+1)(n+2)}$

$= \frac{1}{(n+1)(n+2)}$ **[1 Mark]**

8. By using the properties of definite integrals, evaluate the integral $\int_0^{\frac{\pi}{4}} \log(1+\tan x)\, dx$ **[4 Marks]**

Solution:

Let $I = \int_0^{\frac{\pi}{4}} \log(1+\tan x)\, dx$ ...(i)

$\therefore I = \int_0^{\frac{\pi}{4}} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right] dx \quad \left(\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$ **[1 Mark]**

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log\left\{1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right\} dx$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log\left\{1+\frac{1-\tan x}{1+\tan x}\right\} dx$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log\frac{2}{(1+\tan x)} dx$ **[1 Mark]**

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log 2\, dx - \int_0^{\frac{\pi}{4}} \log(1+\tan x)\, dx$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log 2 dx - I$ [From ...(i)] **[1 Mark]**

$\Rightarrow 2I = [x \log 2]_0^{\frac{\pi}{4}}$

$\Rightarrow 2I = \frac{\pi}{4} \log 2$

$\Rightarrow I = \frac{\pi}{8} \log 2$ **[1 Mark]**

9. By using the properties of definite integrals, evaluate the integral $\int_0^2 x\sqrt{2-x}dx$ **[2 Marks]**

Solution:

Let $I = \int_0^2 x\sqrt{2-x}dx$

$I = \int_0^2 (2-x)\sqrt{x}dx \quad \left(\because \int_0^a f(x)dx \int_0^a f(a-x)\,dx\right)$

$= \int_0^2 \left\{2x^{\frac{1}{2}} - x^{\frac{3}{2}}\right\} dx$

$= \left[2\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) - \frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^2$ **[1 Mark]**

$= \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right]_0^2$

$= \frac{4}{3}(2)^{\frac{3}{2}} - \frac{2}{5}(2)^{\frac{5}{2}}$

$= \frac{4\times2\sqrt{2}}{3} - \frac{2}{5} \times 4\sqrt{2}$

$= \frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5}$

$= \frac{40\sqrt{2}-24\sqrt{2}}{15}$

$= \frac{16\sqrt{2}}{15}$ **[1 Mark]**

10. By using the properties of definite integrals, evaluate the integral $\int_0^{\frac{\pi}{2}}(2\log\sin x - \log\sin 2x)dx$ **[4 Marks]**

Solution:

Let $I = \int_0^{\frac{\pi}{2}}(2\log\sin x - \log\sin 2x)dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}}\{2\log\sin x - \log(2\sin x\cos x)\}dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}}\{2\log\sin x - \log\sin x - \log\cos x - \log 2\}dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}}\{\log\sin x - \log\cos x - \log 2\}dx$ ...(i) **[1 Mark]**

As we know that, $\left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$

$\Rightarrow I = \int_0^{\frac{\pi}{2}}\left\{\log\sin\left(\frac{\pi}{2}-x\right) - \log\cos\left(\frac{\pi}{2}-x\right) - \log 2\right\}dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}}\{\log\cos x - \log\sin x - \log 2\}dx$ ...(ii) **[1 Mark]**

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}}(-\log 2 - \log 2)dx$$

$$\Rightarrow 2I = -2\log 2\int_0^{\frac{\pi}{2}} 1\cdot dx$$

$$\Rightarrow 2I = -2\log 2\,[x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow I = -\log 2\left[\frac{\pi}{2}\right]$$

$$\Rightarrow I = \frac{\pi}{2}(-\log 2)$$

$$\Rightarrow I = \frac{\pi}{2}\left[\log\frac{1}{2}\right]$$

$\Rightarrow I = \frac{\pi}{2}\log\frac{1}{2}$ **[2 Marks]**

11. By using the properties of definite integrals, evaluate the integral $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\, dx$ **[2 Marks]**

Solution:

Let $I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^2 x dx$

Since, $\sin^2(-x) = (\sin(-x))^2 = (-\sin x)^2 = \sin^2 x$, hence, $\sin^2 x$ is an even function.

As we know that if $f(x)$ is an even function, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)\, dx$

Hence, $I = 2\int_0^{\frac{\pi}{2}} \sin^2 x dx$

$= 2\int_0^{\frac{\pi}{2}} \frac{1-\cos 2x}{2} dx$ **[1 Mark]**

$$= \int_0^{\frac{\pi}{2}} (1 - \cos 2x)\, dx$$

$$= \left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} - 0 - 0 + 0$$

$= \frac{\pi}{2}$ **[1 Mark]**

12. By using the properties of definite integrals, evaluate the integral $\int_0^{\pi} \frac{x\, dx}{1+\sin x}$ **[4 Marks]**

Solution:

Let $I = \int_0^{\pi} \frac{x\, dx}{1+\sin x}$ ...(i)

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{1 + \sin(\pi - x)} dx \quad \left(\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{1+\sin x} dx$ ...(ii) **[2 Marks]**

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{\pi}{1 + \sin x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{(1 - \sin x)}{(1 + \sin x)(1 - \sin x)} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{1 - \sin x}{\cos^2 x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \{\sec^2 x - \tan x \sec x\} dx$$

$$\Rightarrow 2I = \pi[\tan x - \sec x]_0^{\pi} = \pi[\tan \pi - \sec \pi - \tan 0 + \sec 0]$$

$$\Rightarrow 2I = \pi[2]$$

$\Rightarrow I = \pi$ **[2 Marks]**

13. By using the properties of definite integrals, evaluate the integral $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \, dx$ **[2 Marks]**

Solution:

Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx$ ...(i)

Since, $\sin^7(-x) = (\sin(-x))^7 = (-\sin x)^7 = -\sin^7 x$, hence, $\sin^2 x$ is an odd function. **[1 Mark]**

As we know that, if $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)dx = 0$

$\therefore I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^7 x\,dx=0$ **[1 Mark]**

14. By using the properties of definite integrals, evaluate the integral $\int_0^{2\pi}\cos^5 x\,dx$ **[4 Marks]**

Solution:

Let $I=\int_0^{2\pi}\cos^5 x dx$ ...(i)

$\cos^5(2\pi-x)=\cos^5 x$ **[1 Mark]**

As we know that,

$\int_0^{2\pi}f(x)dx=2\int_0^a f(x)dx$, if $f(2a-x)=f(x)$ **[1 Mark]**

$=0$ if $f(2a-x)=-f(x)$

$\therefore I=2\int_0^{\pi}\cos^5 xdx=2\int_0^{2\times\frac{\pi}{2}}\cos^5 xdx$ **[1 Mark]**

Again, using the same property, we have

$\Rightarrow I=2(0)=0 \quad [\because \cos^5(\pi-x)=-\cos^5 x]$ **[1 Mark]**

15. By using the properties of definite integrals, evaluate the integral $\int_0^{\frac{\pi}{2}}\frac{\sin x-\cos x}{1+\sin x\cos x}dx$ **[2 Marks]**

Solution:

Let $I=\int_0^{\frac{\pi}{2}}\frac{\sin x-\cos x}{1+\sin x\cos x}dx$ ...(i)

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\sin\left(\frac{\pi}{2}-x\right)-\cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}dx \quad \left(\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1 + \sin x \cos x} dx$ ...(ii) **[1 Mark]**

Adding (i) and (ii), we get

$2I = \int_0^{\frac{\pi}{2}} \frac{0}{1 + \sin x \cos x} dx$

$\Rightarrow I = 0$ **[1 Mark]**

16. By using the properties of definite integrals, evaluate the integral $\int_0^{\pi} \log(1 + \cos x)\, dx$ **[6 Marks]**

Solution:

Let $I = \int_0^{\pi} \log(1 + \cos x)\, dx$ ...(i)

$$\Rightarrow I = \int_0^{\pi} \log(1 + \cos(\pi - x))\, dx \quad \left(\because \int_0^a f(x) dx = \int_0^a f(a - x)\, dx\right)$$

$\Rightarrow I = \int_0^{\pi} \log(1 - \cos x)\, dx$ ...(ii) **[1 Mark]**

Adding (i) and (ii), we get

$2I = \int_0^{\pi} \{\log(1 + \cos x) + \log(1 - \cos x)\} dx$

$\Rightarrow 2I = \int_0^{\pi} \log(1 - \cos^2 x)\, dx$

$\Rightarrow 2I = \int_0^{\pi} \log \sin^2 x\, dx$

$\Rightarrow 2I = 2\int_0^{\pi} \log \sin x\, dx$

$\Rightarrow I = \int_0^{\pi} \log \sin x\, dx$ ...(iii) **[1 Mark]**

Since, $\sin(\pi - x) = \sin x$

$\therefore I = 2\int_0^{\frac{\pi}{2}} \log \sin x\, dx$ ...(iv) $\quad [\int_0^{2\pi} f(x) dx = 2\int_0^{a} f(x) dx$, if $f(2a - x) = f(x)]$

$\Rightarrow I = 2\int_0^{\frac{\pi}{2}} \log\sin\left(\frac{\pi}{2} - x\right) dx = 2\int_0^{\frac{\pi}{2}} \log\cos x\, dx$ ...(v) **[1 Mark]**

Adding (iv) and (v), we get

$2I = 2\int_0^{\frac{\pi}{2}} (\log\sin x + \log\cos x) dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} (\log\sin x + \log\cos x + \log 2 - \log 2) dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} (\log 2\sin x\cos x - \log 2) dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log\sin 2x\, dx - \int_0^{\frac{\pi}{2}} \log 2\, dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log\sin 2x\, dx - \log 2\, [x]_0^{\frac{\pi}{2}}$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log\sin 2x\, dx - \frac{\pi}{2}\log 2$ **[2 Marks]**

Let $2x = t \Rightarrow 2dx = dt$

When $x = 0, t = 0$, when $x = \frac{\pi}{2}, t = \pi$

$\therefore I = \frac{1}{2}\int_0^{\pi} \log\sin t\, dt - \frac{\pi}{2}\log 2$

$\therefore I = \frac{1}{2}\int_0^{\pi} \log\sin x\, dx - \frac{\pi}{2}\log 2$ $\quad [\because \int_a^b f(x)dx = \int_a^b f(t)dt]$

$\Rightarrow I = \frac{1}{2}I - \frac{\pi}{2}\log 2$ [from (iii)]

$\Rightarrow \frac{I}{2} = -\frac{\pi}{2}\log 2$

$\Rightarrow I = -\pi\log 2$ **[1 Mark]**

17. By using the properties of definite integrals, evaluate the integral $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}} dx$ **[2 Marks]**

Solution:

Let $I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}} dx$ ...(i)

As we know that, $\left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$

Hence, $I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}} dx$ ...(ii) **[1 Mark]**

Adding (i) and (ii), we get

$2I = \int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}} dx$

$\Rightarrow 2I = \int_0^a 1 \cdot dx$

$\Rightarrow 2I = [x]_0^a$

$\Rightarrow 2I = a$

$\Rightarrow I = \frac{a}{2}$ **[1 Mark]**

18. By using the properties of definite integrals, evaluate the integral $\int_0^4 |x-1| dx$ **[4 Marks]**

Solution:

$I = \int_0^4 |x-1| dx$

It is seen that, $(x-1) \leq 0$ when $0 \leq x \leq 1$ and $(x-1) \geq 0$ when $1 \leq x \leq 4$ **[1 Mark]**

Hence, $I = \int_0^1 |x-1| dx + \int_1^4 |x-1| dx \quad \left(\because \int_0^4 f(x)dx = \int_0^1 f(x)dx + \int_1^4 f(x)\,dx\right)$ **[1 Mark]**

$= \int_0^1 -(x-1)dx + \int_1^4 (x-1)dx$

$= \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^4$

$= 1 - \frac{1}{2} + \frac{(4)^2}{2} - 4 - \frac{1}{2} + 1$

$= 1 - \frac{1}{2} + 8 - 4 - \frac{1}{2} + 1$

$= 5$ **[2 Marks]**

19. Show that $\int_0^a f(x)g(x)dx = 2\int_0^a f(x)dx$, if f and g are defined as $f(x) = f(a-x)$ and $g(x) + g(a-x) = 4$ **[4 Marks]**

Solution:

Let $I = \int_0^a f(x)g(x)dx$ ...(i)

$\Rightarrow I = \int_0^a f(a-x)g(a-x)dx \quad \left(\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$

$\Rightarrow I = \int_0^a f(x)g(a-x)dx$ ...(ii) **[2 Marks]**

Adding (i) and (ii), we get

$2I = \int_0^a \{f(x)g(x) + f(x)g(a-x)\}dx$

$\Rightarrow 2I = \int_0^a f(x)\{g(x) + g(a-x)\}dx$

$\Rightarrow 2I = \int_0^a f(x) \times 4dx \quad [\because g(x) + g(a-x) = 4]$

$\Rightarrow I = 2\int_0^a f(x)dx$

Hence, $\int_0^a f(x)g(x)dx = 2\int_0^a f(x)dx$ **[2 Marks]**

20. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x + \tan^5 x + 1)dx$ is **[4 Marks]**

(A) 0

(B) 2

(C) π

(D) 1

Solution:

Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x + \tan^5 x + 1)dx$

$\Rightarrow I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos x + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^5 x dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \cdot dx$ **[1 Mark]**

As we know that if $f(x)$ is an even function, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

And if $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)dx = 0$ **[1 Mark]**

Here, $x^3, x\cos x$, and $\tan^5 x$ are odd functions and 1 is an even function.

Hence, $I = 0 + 0 + 0 + 2\int_0^{\frac{\pi}{2}} 1 \cdot dx$

$= 2[x]_0^{\frac{\pi}{2}}$

$= \frac{2\pi}{2} = \pi$

Hence, (C) is the correct answer. **[2 Marks]**

21. The value of $\int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx$ is **[4 Marks]**

(A) 2

(B) $\frac{3}{4}$

(C) 0

(D) -2

Solution:

Let $I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx$ ...(i)

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log\left[\frac{4+3\sin\left(\frac{\pi}{2}-x\right)}{4+3\cos\left(\frac{\pi}{2}-x\right)}\right] dx \quad \left(\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\cos x}{4+3\sin x}\right) dx$ ...(ii) **[2 Marks]**

Adding (i) and (ii), we get

$2I = \int_0^{\frac{\pi}{2}} \left\{ \log\left(\frac{4+3\sin x}{4+3\cos x}\right) + \log\left(\frac{4+3\cos x}{4+3\sin x}\right) \right\} dx$

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x} \times \frac{4+3\cos x}{4+3\sin x}\right) dx$

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log 1 dx$

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 0\, dx$

$\Rightarrow I = 0$

Hence, (C) is the correct answer. **[2 Marks]**

Miscellaneous Exercise on Chapter 7

1. Integrate the function $\frac{1}{x-x^3}$ **[4 Marks]**

Solution:

We need to integrate $\frac{1}{x-x^3}$

$\int \frac{1}{x-x^3} dx = \int \frac{1}{x^3\left(\frac{1}{x^2}-1\right)} dx$ **[1 Mark]**

Let $\frac{1}{x^2} - 1 = t$

$\Rightarrow -\frac{2}{x^3} dx = dt$

$\Rightarrow \frac{dx}{x^3} = -\frac{1}{2} dt$ **[1 Mark]**

Hence, $\int \frac{1}{x-x^3} dx = -\frac{1}{2} \int \frac{1}{t} dt$

$= -\frac{1}{2} \log|t| + C$

$= -\frac{1}{2} \log\left|\frac{1}{x^2} - 1\right| + C$

$= -\frac{1}{2} \log\left|\frac{1-x^2}{x^2}\right| + C$

$= \frac{1}{2}\log\left|\frac{x^2}{1-x^2}\right| + C$ **[2 Marks]**

2. Integrate the function $\frac{1}{\sqrt{x+a}+\sqrt{x+b}}$ **[4 Marks]**

Solution:

We need to integrate $\frac{1}{\sqrt{x+a}+\sqrt{x+b}}$

$\frac{1}{\sqrt{x+a}+\sqrt{x+b}} = \frac{1}{\sqrt{x+a}+\sqrt{x+b}} \times \frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}$

$= \frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)}$

$= \frac{\left(\sqrt{x+a}-\sqrt{x+b}\right)}{a-b}$ **[2 Marks]**

$\Rightarrow \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}\, dx = \frac{1}{a-b}\int\left(\sqrt{x+a} - \sqrt{x+b}\right) dx$

$= \frac{1}{(a-b)}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}\right] + C$

$= \frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}}\right] + C$ **[2 Marks]**

3. Integrate the function $\frac{1}{x\sqrt{ax-x^2}}$ $\left[\text{Hint: Put } x = \frac{a}{t}\right]$ **[4 Marks]**

Solution:

We need to integrate $\frac{1}{x\sqrt{ax-x^2}}$

Let $x = \frac{a}{t} \Rightarrow dx = -\frac{a}{t^2}dt$

$\Rightarrow \int \frac{1}{x\sqrt{ax-x^2}}dx = \int \frac{1}{\frac{a}{t}\sqrt{a\cdot\frac{a}{t}-\left(\frac{a}{t}\right)^2}}\left(-\frac{a}{t^2}dt\right)$ **[1 Mark]**

$= -\int \frac{1}{at}\cdot\frac{1}{\sqrt{\frac{1}{t}-\frac{1}{t^2}}}dt$

$= -\frac{1}{a}\int \frac{1}{\sqrt{\frac{t^2}{t}-\frac{t^2}{t^2}}}dt$ **[1 Mark]**

$= -\frac{1}{a}\int \frac{1}{\sqrt{t-1}}dt$

$= -\frac{1}{a}\left[2\sqrt{t-1}\right] + C$ **[1 Mark]**

$= -\frac{1}{a}\left[2\sqrt{\frac{a}{x}-1}\right] + C$

$= -\frac{2}{a}\sqrt{\frac{a-x}{x}} + C$ **[1 Mark]**

4. Integrate the function $\frac{1}{x^2(x^4+1)^{\frac{3}{4}}}$ **[4 Marks]**

Solution:

We need to integrate $\frac{1}{x^2(x^4+1)^{\frac{3}{4}}}$

Multiplying and dividing by x^{-3}, we get

$\frac{x^{-3}}{x^2\cdot x^{-3}(x^4+1)^{\frac{3}{4}}} = \frac{x^{-3}(x^4+1)^{\frac{-3}{4}}}{x^2\cdot x^{-3}}$ **[1 Mark]**

$= \frac{(x^4+1)^{\frac{-3}{4}}}{x^5\cdot(x^4)^{-\frac{3}{4}}}$

$= \frac{1}{x^5}\left(\frac{x^4+1}{x^4}\right)^{-\frac{3}{4}}$

$= \frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}$ [1 Mark]

Let $\frac{1}{x^4} = t \Rightarrow -\frac{4}{x^5}dx = dt \Rightarrow \frac{1}{x^5}dx = -\frac{dt}{4}$

$\therefore \int \frac{1}{x^2(x^4+1)^{\frac{3}{4}}}dx = \int \frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}dx$ [1 Mark]

$= -\frac{1}{4}\int (1+t)^{-\frac{3}{4}}dt$

$= -\frac{1}{4}\left[\frac{(1+t)^{\frac{1}{4}}}{\frac{1}{4}}\right] + C$

$= -\frac{1}{4}\frac{\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}}{\frac{1}{4}} + C$

$= -\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}} + C$ [1 Mark]

5. Integrate the function $\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}$ $\left[\text{Hint: } \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}} = \frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}, \text{ put } x = t^6\right]$ [4 Marks]

Solution:

We need to integrate $\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}} = \frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}$

Let $x = t^6 \Rightarrow dx = 6t^5dt$

$\therefore \int \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx = \int \frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}dx$ [1 Mark]

$= \int \frac{6t^5}{t^2(1+t)}dt$

$= 6\int \frac{t^3}{(1+t)}dt$ [1 Mark]

On dividing, we get

$\int \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}} dx = 6\int \left\{(t^2 - t + 1) - \frac{1}{1+t}\right\} dt$

$= 6\left[\left(\frac{t^3}{3}\right) - \left(\frac{t^2}{2}\right) + t - \log|1+t|\right]$

$= 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6\log\left(1 + x^{\frac{1}{6}}\right) + C$

$= 2\sqrt{x} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6\log\left(1 + x^{\frac{1}{6}}\right) + C$ **[2 Marks]**

6. Integrate the function $\frac{5x}{(x+1)(x^2+9)}$ **[4 Marks]**

Solution:

We need to integrate $\frac{5x}{(x+1)(x^2+9)}$

Let $\frac{5x}{(x+1)(x^2+9)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+9)}$ ...(i)

$\Rightarrow 5x = A(x^2+9) + (Bx+C)(x+1)$

$\Rightarrow 5x = Ax^2 + 9A + Bx^2 + Bx + Cx + C$ **[1 Mark]**

On equating the coefficients of x^2, x and constant term, we get

$A + B = 0$

$B + C = 5$

$9A + C = 0$

After solving these equations, we get

$A = -\frac{1}{2}, B = \frac{1}{2}$ and $C = \frac{9}{2}$ **[1 Mark]**

From equation (i), we get

$\frac{5x}{(x+1)(x^2+9)} = \frac{-1}{2(x+1)} + \frac{\frac{x}{2}+\frac{9}{2}}{(x^2+9)}$

$\int \frac{5x}{(x+1)(x^2+9)} dx = \int \left\{\frac{-1}{2(x+1)} + \frac{(x+9)}{2(x^2+9)}\right\} dx$

$= -\frac{1}{2}\log|x+1| + \frac{1}{2}\int \frac{x}{x^2+9}dx + \frac{9}{2}\int \frac{1}{x^2+9}dx$

$= -\frac{1}{2}\log|x+1| + \frac{1}{4}\int \frac{2x}{x^2+9}dx + \frac{9}{2}\int \frac{1}{x^2+9}dx$

$= -\frac{1}{2}\log|x+1| + \frac{1}{4}\log|x^2+9| + \frac{9}{2}.\frac{1}{3}\tan^{-1}\frac{x}{3} + C$

$= -\frac{1}{2}\log|x+1| + \frac{1}{4}\log(x^2+9) + \frac{3}{2}\tan^{-1}\frac{x}{3} + C$ **[2 Marks]**

7. Integrate the function $\frac{\sin x}{\sin(x-a)}$ **[4 Marks]**

Solution:

We need to integrate $\frac{\sin x}{\sin(x-a)}$

Let $x - a = t \Rightarrow dx = dt$

$$\int \frac{\sin x}{\sin(x-a)}dx = \int \frac{\sin(t+a)}{\sin t}dt$$

$$= \int \frac{\sin t \cos a + \cos t \sin a}{\sin t}dt$$

$= \int(\cos a + \cot t \sin a)dt$ **[2 Marks]**

$= t\cos a + \sin a \log|\sin t| + C_1$

$= (x-a)\cos a + \sin a \log|\sin(x-a)| + C_1$

$= x\cos a + \sin a \log|\sin(x-a)| - a\cos a + C_1$

$= \sin a \log|\sin(x-a)| + x\cos a + C$ **[2 Marks]**

8. Integrate the function $\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$ **[2 Marks]**

Solution:

We need to integrate $\frac{e^{5\log x}-e^{4\log x}}{e^{3\log x}-e^{2\log x}}$

$\frac{e^{5\log x}-e^{4\log x}}{e^{3\log x}-e^{2\log x}} = \frac{e^{4\log x}\left(e^{\log x}-1\right)}{e^{2\log x}\left(e^{\log x}-1\right)}$ **[1 Mark]**

$= e^{2\log x}$

$= e^{\log x^2}$

$= x^2$

$\therefore \int \frac{e^{5\log x}-e^{4\log x}}{e^{3\log x}-e^{2\log x}} dx = \int x^2 dx = \frac{x^3}{3} + C$ **[1 Mark]**

9. Integrate the function $\frac{\cos x}{\sqrt{4-\sin^2 x}}$ **[2 Marks]**

Solution:

We need to integrate $\frac{\cos x}{\sqrt{4-\sin^2 x}}$

Let $\sin x = t \Rightarrow \cos x\, dx = dt$

$\Rightarrow \int \frac{\cos x}{\sqrt{4-\sin^2 x}} dx = \int \frac{dt}{\sqrt{(2)^2-(t)^2}}$ **[1 Mark]**

$= \sin^{-1}\left(\frac{t}{2}\right) + C$

$= \sin^{-1}\left(\frac{\sin x}{2}\right) + C$ **[1 Mark]**

10. Integrate the function $\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}$ **[2 Marks]**

Solution:

We need to integrate $\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}$

$\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}=\frac{(\sin^4 x+\cos^4 x)(\sin^4 x-\cos^4 x)}{\sin^2 x+\cos^2 x-\sin^2 x\cos^2 x-\sin^2 x\cos^2 x}$ **[1 Mark]**

$=\frac{(\sin^4 x+\cos^4 x)(\sin^2 x+\cos^2 x)(\sin^2 x-\cos^2 x)}{(\sin^2 x-\sin^2 x\cos^2 x)+(\cos^2 x-\sin^2 x\cos^2 x)}$

$=\frac{(\sin^4 x+\cos^4 x)(\sin^2 x-\cos^2 x)}{\sin^2 x(1-\cos^2 x)+\cos^2 x(1-\sin^2 x)}$

$=\frac{-(\sin^4 x+\cos^4 x)(\cos^2 x-\sin^2 x)}{(\sin^4 x+\cos^4 x)}$

$=-\cos 2x$

$\therefore \int \frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}dx=\int -\cos 2x\, dx=-\frac{\sin 2x}{2}+C$ **[1 Mark]**

11. Integrate the function $\frac{1}{\cos(x+a)\cos(x+b)}$ **[4 Marks]**

Solution:

We need to integrate $\frac{1}{\cos(x+a)\cos(x+b)}$

Multiplying and dividing by $\sin(a-b)$, we get

$\frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x+a)\cos(x+b)}\right]$

$=\frac{1}{\sin(a-b)}\left[\frac{\sin[(x+a)-(x+b)]}{\cos(x+a)\cos(x+b)}\right]$

$=\frac{1}{\sin(a-b)}\left[\frac{\sin(x+a)\cos(x+b)-\cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)}\right]$

$=\frac{1}{\sin(a-b)}\left[\frac{\sin(x+a)}{\cos(x+a)}-\frac{\sin(x+b)}{\cos(x+b)}\right]$

$=\frac{1}{\sin(a-b)}[\tan(x+a)-\tan(x+b)]$ **[2 Marks]**

$\int\frac{1}{\cos(x+a)\cos(x+b)}dx=\frac{1}{\sin(a-b)}\int[\tan(x+a)-\tan(x+b)]\,dx$

$=\frac{1}{\sin(a-b)}[-\log|\cos(x+a)|+\log|\cos(x+b)|]+C$

$=\frac{1}{\sin(a-b)}\log\left|\frac{\cos(x+b)}{\cos(x+a)}\right|+C$ **[2 Marks]**

12. Integrate the function $\frac{x^3}{\sqrt{1-x^8}}$ **[2 Marks]**

Solution:

We need to integrate $\frac{x^3}{\sqrt{1-x^8}}$

Let $x^4 = t \Rightarrow 4x^3 dx = dt$

$\Rightarrow \int \frac{x^3}{\sqrt{1-x^8}} dx = \frac{1}{4} \int \frac{dt}{\sqrt{1-t^2}}$ **[1 Mark]**

$= \frac{1}{4} \sin^{-1} t + C$

$= \frac{1}{4} \sin^{-1}(x^4) + C$ **[1 Mark]**

13. Integrate the function $\frac{e^x}{(1+e^x)(2+e^x)}$ **[4 Marks]**

Solution:

We need to integrate $\frac{e^x}{(1+e^x)(2+e^x)}$

Let $e^x = t \Rightarrow e^x dx = dt$

$$\Rightarrow \int \frac{e^x}{(1+e^x)(2+e^x)} dx = \int \frac{dt}{(t+1)(t+2)}$$

$$= \int \frac{(t+2)-(t+1)}{(t+1)(t+2)} dt$$

$= \int \left[\frac{1}{(t+1)} - \frac{1}{(t+2)}\right] dt$ **[2 Marks]**

$= \log|t+1| - \log|t+2| + C$

$$= \log\left|\frac{t+1}{t+2}\right| + C$$

$= \log\left|\frac{1+e^x}{2+e^x}\right| + C$ **[2 Marks]**

14. Integrate the function $\frac{1}{(x^2+1)(x^2+4)}$ **[4 Marks]**

Solution:

We need to integrate $\frac{1}{(x^2+1)(x^2+4)}$

$\therefore \frac{1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+4)}$

$\Rightarrow 1 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$

$\Rightarrow 1 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Cx + Dx^2 + D$ **[1 Mark]**

On equating the coefficients of x^3, x^2, x and constant term, we get

$A + C = 0$

$B + D = 0$

$4A + C = 0$

$4B + D = 1$ **[1 Mark]**

After solving these equations, we get

$A = 0, B = \frac{1}{3}, C = 0$ and $D = -\frac{1}{3}$ **[1 Mark]**

From equation (i), we get

$\frac{1}{(x^2+1)(x^2+4)} = \frac{1}{3(x^2+1)} - \frac{1}{3(x^2+4)}$

$\int \frac{1}{(x^2+1)(x^2+4)} dx = \frac{1}{3}\int \frac{1}{x^2+1} dx - \frac{1}{3}\int \frac{1}{x^2+4} dx$

$= \frac{1}{3}\tan^{-1} x - \frac{1}{3}\cdot\frac{1}{2}\tan^{-1}\frac{x}{2} + C$

$= \frac{1}{3}\tan^{-1} x - \frac{1}{6}\tan^{-1}\frac{x}{2} + C$ **[1 Mark]**

15. Integrate the function $\cos^3 x\, e^{\log \sin x}$ **[2 Marks]**

Solution:

We need to integrate $\cos^3 x\, e^{\log \sin x} = \cos^3 x \sin x$

Let $\cos x = t \Rightarrow -\sin x\, dx = dt$

$\Rightarrow \int \cos^3 x e^{\log \sin x} dx = \int \cos^3 x \sin x dx$ **[1 Mark]**

$= -\int t^3 dt$

$= -\frac{t^4}{4} + C$

$= -\frac{\cos^4 x}{4} + C$ **[1 Mark]**

16. Integrate the function $e^{3\log x}(x^4+1)^{-1}$ **[2 Marks]**

Solution:

We need to integrate $e^{3\log x}(x^4+1)^{-1} = e^{\log x^3}(x^4+1)^{-1} = \frac{x^3}{(x^4+1)}$

Let $x^4 + 1 = t \Rightarrow 4x^3 dx = dt$

$\Rightarrow \int e^{3\log x}(x^4+1)^{-1} dx = \int \frac{x^3}{(x^4+1)} dx$

$= \frac{1}{4}\int \frac{dt}{t}$ **[1 Mark]**

$= \frac{1}{4}\log|t| + C$

$= \frac{1}{4}\log|x^4+1| + C$

$= \frac{1}{4}\log(x^4+1) + C$ **[1 Mark]**

17. Integrate the function $f'(ax+b)[f(ax+b)]^n$ **[2 Marks]**

Solution:

We need to integrate $f'(ax+b)[f(ax+b)]^n$

Let $f(ax+b) = t \Rightarrow af'(ax+b)dx = dt$

$\Rightarrow \int f'(ax+b)\,[f(ax+b)]^n dx = \frac{1}{a}\int t^n dt$ **[1 Mark]**

$= \frac{1}{a}\left[\frac{t^{n+1}}{n+1}\right] + C$

$= \frac{1}{a(n+1)}\big(f(ax+b)\big)^{n+1} + C$ **[1 Mark]**

18. Integrate the function $\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$ **[4 Marks]**

Solution:

We need to integrate $\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$

$\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} = \frac{1}{\sqrt{\sin^3 x(\sin x \cos\alpha + \cos x \sin\alpha)}}$

$= \frac{1}{\sqrt{\sin^4 x \cos\alpha + \sin^3 x \cos x \sin\alpha}}$

$= \frac{1}{\sin^2 x\sqrt{\cos\alpha + \cot x \sin\alpha}}$

$= \frac{\operatorname{cosec}^2 x}{\sqrt{\cos\alpha + \cot x \sin\alpha}}$ **[2 Marks]**

Let $\cos\alpha + \cot x \sin\alpha = t \Rightarrow -\operatorname{cosec}^2 x \sin\alpha\, dx = dt$

$\therefore \int \frac{1}{\sin^3 x \sin(x+\alpha)} dx = \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cos\alpha + \cot x \sin\alpha}} dx$

$= \frac{-1}{\sin\alpha}\int \frac{dt}{\sqrt{t}}$

$= \frac{-1}{\sin\alpha}\left[2\sqrt{t}\right] + C$

$= \frac{-1}{\sin\alpha}\left[2\sqrt{\cos\alpha + \cot x \sin\alpha}\right] + C$

$= \frac{-2}{\sin\alpha}\sqrt{\cos\alpha + \frac{\cos x \sin\alpha}{\sin x}} + C$

$= \frac{-2}{\sin\alpha}\sqrt{\frac{\sin x \cos\alpha + \cos x \sin\alpha}{\sin x}} + C$

$= -\frac{2}{\sin\alpha}\sqrt{\frac{\sin(x+\alpha)}{\sin x}} + C$ **[2 Marks]**

19. Integrate the function $\frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}, x \in [0, 1]$ **[6 Marks]**

Solution:

We need to integrate $\frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}, x \in [0, 1]$

Let $I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}} dx$

As we know that, $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

Hence, $\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \frac{\pi}{2}$ **[1 Mark]**

$\Rightarrow I = \int \frac{\left(\frac{\pi}{2} - \cos^{-1}\sqrt{x}\right) - \cos^{-1}\sqrt{x}}{\frac{\pi}{2}} dx$

$= \frac{2}{\pi}\int \left(\frac{\pi}{2} - 2\cos^{-1}\sqrt{x}\right) dx$

$= \frac{2}{\pi}\cdot\frac{\pi}{2}\int 1 \cdot dx - \frac{4}{\pi}\int \cos^{-1}\sqrt{x}\, dx$

$= x - \frac{4}{\pi}\int \cos^{-1}\sqrt{x}\, dx$ ...(i) **[1 Mark]**

Let $I_1 = \int \cos^{-1}\sqrt{x}\ dx$

Also, consider $x = t^2 \Rightarrow dx = 2t\, dt$

$\Rightarrow I_1 = 2\int \cos^{-1} t \cdot t\, dt$

$= 2\left[\cos^{-1} t \cdot \frac{t^2}{2} - \int \frac{-1}{\sqrt{1-t^2}} \cdot \frac{t^2}{2} dt\right]$ (using integration by parts) **[1 Mark]**

$= t^2 \cos^{-1} t + \int \frac{t^2}{\sqrt{1-t^2}} dt$

$= t^2 \cos^{-1} t - \int \frac{1-t^2-1}{\sqrt{1-t^2}} dt$

$= t^2 \cos^{-1} t - \int \sqrt{1-t^2} dt + \int \frac{1}{\sqrt{1-t^2}} dt$

$= t^2 \cos^{-1} t - \frac{t}{2}\sqrt{1-t^2} - \frac{1}{2}\sin^{-1} t + \sin^{-1} t$

$= t^2 \cos^{-1} t - \frac{t}{2}\sqrt{1-t^2} + \frac{1}{2}\sin^{-1} t$

$= x \cos^{-1}\sqrt{x} - \frac{\sqrt{x}}{2}\sqrt{1-x} + \frac{1}{2}\sin^{-1}\sqrt{x}$ **[1 Mark]**

From equation (i), we get

$I = x - \frac{4}{\pi}\left[x \cos^{-1}\sqrt{x} - \frac{\sqrt{x}}{2}\sqrt{1-x} + \frac{1}{2}\sin^{-1}\sqrt{x}\right]$

$= x - \frac{4}{\pi}\left[x\left(\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right) - \frac{\sqrt{x-x^2}}{2} + \frac{1}{2}\sin^{-1}\sqrt{x}\right]$

$= x - 2x + \frac{4x}{\pi}\sin^{-1}\sqrt{x} + \frac{2}{\pi}\sqrt{x-x^2} - \frac{2}{\pi}\sin^{-1}\sqrt{x}$

$= -x + \frac{2}{\pi}\left[(2x-1)\sin^{-1}\sqrt{x}\right] + \frac{2}{\pi}\sqrt{x-x^2} + C$

$= \frac{2(2x-1)}{\pi}\sin^{-1}\sqrt{x} + \frac{2}{\pi}\sqrt{x-x^2} - x + C$ **[2 Marks]**

20. Integrate the function $\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$ **[6 Marks]**

Solution:

We need to integrate $\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$

Let $I = \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} dx$

Again, let $x = \cos^2\theta \Rightarrow dx = -2\sin\theta\cos\theta d\theta$ **[1 Mark]**

$I=\int\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(-2\sin\theta\cos\theta)d\theta$

$=-\int\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}2\sin\theta\cos\theta\, d\theta$

$=-2\int\tan\frac{\theta}{2}\cdot\sin\theta\cos\theta d\theta$

$=-2\int\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\cos\theta d\theta$

$=-4\int\sin^2\frac{\theta}{2}\cos\theta d\theta$

$=-4\int\sin^2\frac{\theta}{2}\cdot\left(2\cos^2\frac{\theta}{2}-1\right)d\theta$

$=-4\int\left(2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)d\theta$

$=-8\int\sin^2\frac{\theta}{2}\cdot\cos^2\frac{\theta}{2}d\theta+4\int\sin^2\frac{\theta}{2}d\theta$

$=-2\int\sin^2\theta d\theta+4\int\sin^2\frac{\theta}{2}d\theta$

$=-2\int\left(\frac{1-\cos 2\theta}{2}\right)d\theta+4\int\frac{1-\cos\theta}{2}d\theta$ **[2 Marks]**

$=-2\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]+4\left[\frac{\theta}{2}-\frac{\sin\theta}{2}\right]+C$

$=-\theta+\frac{\sin 2\theta}{2}+2\theta-2\sin\theta+C$

$=\theta+\frac{\sin 2\theta}{2}-2\sin\theta+C$

$=\theta+\frac{2\sin\theta\cos\theta}{2}-2\sin\theta+C$

$=\theta+\sqrt{1-\cos^2\theta}\cdot\cos\theta-2\sqrt{1-\cos^2\theta}+C$

$=\cos^{-1}\sqrt{x}+\sqrt{1-x}\cdot\sqrt{x}-2\sqrt{1-x}+C$

$=-2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x(1-x)}+C$

$=-2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x-x^2}+C$ **[3 Marks]**

21. Integrate the function $\frac{2+\sin 2x}{1+\cos 2x}e^x$ **[4 Marks]**

Solution:

We need to integrate $\frac{2+\sin 2x}{1+\cos 2x}e^x$

Let, $I=\int\left(\frac{2+\sin 2x}{1+\cos 2x}\right)e^x\,dx$

$=\int\left(\frac{2+2\sin x\cos x}{2\cos^2 x}\right)e^x$

$=\int\left(\frac{1+\sin x\cos x}{\cos^2 x}\right)e^x$

$=\int(\sec^2 x+\tan x)e^x$ **[2 Marks]**

Let $f(x)=\tan x\Rightarrow f'(x)=\sec^2 x$

$\therefore I=\int(f(x)+f'(x)]e^x\,dx$

$=e^x f(x)+C$

$=e^x\tan x+C$ **[2 Marks]**

22. Integrate the function $\frac{x^2+x+1}{(x+1)^2(x+2)}$ **[4 Marks]**

Solution:

We need to integrate $\frac{x^2+x+1}{(x+1)^2(x+2)}$

Let $\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{A}{(x+1)}+\frac{B}{(x+1)^2}+\frac{C}{(x+2)}$ ...(i) **[1 Mark]**

$\Rightarrow x^2+x+1=A(x+1)(x+2)+B(x+2)+C(x^2+2x+1)$

$\Rightarrow x^2+x+1=A(x^2+3x+2)+B(x+2)+C(x^2+2x+1)$

$\Rightarrow x^2+x+1=(A+C)x^2+(3A+B+2C)x+(2A+2B+C)$ **[1 Mark]**

After equating the coefficients of x^2, x and constant term, we get

$A+C=1$

$3A+B+2C=1$

$2A + 2B + C = 1$

After solving these equations, we get

$A = -2, B = 1$ and $C = 3$ **[1 Mark]**

From equation (i), we get

$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{-2}{(x+1)} + \frac{3}{(x+2)} + \frac{1}{(x+1)^2}$

Hence, $\int \frac{x^2+x+1}{(x+1)^2(x+2)} dx = -2\int \frac{1}{x+1} dx + 3\int \frac{1}{(x+2)} dx + \int \frac{1}{(x+1)^2} dx$

$= -2\log|x+1| + 3\log|x+2| - \frac{1}{x+1} + C$ **[1 Mark]**

23. Integrate the function $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$ **[4 Marks]**

Solution:

We need to integrate $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$

Let $I = \int \tan^{-1}\sqrt{\frac{1-x}{1+x}} dx$

Again, let $x = \cos\theta \Rightarrow dx = -\sin\theta d\theta$ **[1 Mark]**

$I = \int \tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(-\sin\theta d\theta)$

$= -\int \tan^{-1}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\sin\theta d\theta$

$= -\int \tan^{-1}\tan\frac{\theta}{2} \cdot \sin\theta d\theta$

$= -\frac{1}{2}\int \theta \cdot \sin\theta d\theta$ **[1 Mark]**

$= -\frac{1}{2}[\theta \cdot (-\cos\theta) - \int 1 \cdot (-\cos\theta) d\theta]$ (using integration by parts)

$= -\frac{1}{2}[-\theta\cos\theta + \sin\theta] + C$

$= \frac{1}{2}\theta\cos\theta - \frac{1}{2}\sin\theta + C$

$= \frac{1}{2}\cos^{-1}x \cdot x - \frac{1}{2}\sqrt{1-x^2} + C$

$= \frac{x}{2}\cos^{-1}x - \frac{1}{2}\sqrt{1-x^2} + C$

$= \frac{1}{2}\left(x\cos^{-1}x - \sqrt{1-x^2}\right) + C$ **[2 Marks]**

24. Integrate the function $\frac{\sqrt{x^2+1}\left[\log(x^2+1)-2\log x\right]}{x^4}$ **[6 Marks]**

Solution:

We need to integrate $\frac{\sqrt{x^2+1}\left[\log(x^2+1)-2\log x\right]}{x^4}$

$\frac{\sqrt{x^2+1}\left[\log(x^2+1)-2\log x\right]}{x^4} = \frac{\sqrt{x^2+1}}{x^4}\left[\log(x^2+1) - \log x^2\right]$

$= \frac{\sqrt{x^2+1}}{x^4}\left[\log\left(\frac{x^2+1}{x^2}\right)\right]$

$= \frac{\sqrt{x^2+1}}{x^4}\log\left(1+\frac{1}{x^2}\right)$

$= \frac{1}{x^3}\sqrt{\frac{x^2+1}{x^2}}\log\left(1+\frac{1}{x^2}\right)$

$= \frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)$ **[2 Marks]**

Let $1+\frac{1}{x^2} = t \Rightarrow \frac{-2}{x^3}dx = dt$

$\therefore I = \int \frac{\sqrt{x^2+1}\left[\log(x^2+1)-2\log x\right]}{x^4}dx$ **[1 Mark]**

$= \int \frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)dx$

$= -\frac{1}{2}\int \sqrt{t}\log t\,dt$

$= -\frac{1}{2}\int t^{\frac{1}{2}} \cdot \log t\,dt$ **[1 Mark]**

Using integration by parts, we get

$I = -\frac{1}{2}\left[\log t \cdot \int t^{\frac{1}{2}} dt - \int \left\{\left(\frac{d}{dt}\log t\right)\int t^{\frac{1}{2}} dt\right\} dt\right]$

$= -\frac{1}{2}\left[\log t \cdot \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \int \frac{1}{t}\cdot\frac{t^{\frac{3}{2}}}{\frac{3}{2}} dt\right]$

$= -\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t - \frac{2}{3}\int t^{\frac{1}{2}} dt\right]$

$= -\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t - \frac{4}{9}t^{\frac{3}{2}}\right] + C$

$= -\frac{1}{3}t^{\frac{3}{2}}\log t + \frac{2}{9}t^{\frac{3}{2}} + C$

$= -\frac{1}{3}t^{\frac{3}{2}}\left[\log t - \frac{2}{3}\right] + C$

$= -\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}\left[\log\left(1+\frac{1}{x^2}\right) - \frac{2}{3}\right] + C$ **[2 Marks]**

25. Evaluate the definite integral $\int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right) dx$ **[4 Marks]**

Solution:

$I = \int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right) dx$

$= \int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right) dx$

$= \int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{\operatorname{cosec}^2\frac{x}{2}}{2} - \cot\frac{x}{2}\right) dx$ **[1 Mark]**

Let $f(x) = -\cot\frac{x}{2}$

$\Rightarrow f'(x) = -\left(-\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}\right) = \frac{1}{2}\operatorname{cosec}^2\frac{x}{2}$ **[1 Mark]**

$\therefore I = \int_{\frac{\pi}{2}}^{\pi} e^x\left(f(x) + f'(x)\right) dx$ **[1 Mark]**

$= [e^x \cdot f(x)]_{\frac{\pi}{2}}^{\pi}$

$= -\left[e^x \cdot \cot\frac{x}{2}\right]_{\frac{\pi}{2}}^{\pi}$

$= -\left[e^{\pi} \times \cot\frac{\pi}{2} - e^{\frac{\pi}{2}} \times \cot\frac{\pi}{4}\right]$

$= -\left[e^{\pi} \times 0 - e^{\frac{\pi}{2}} \times 1\right]$

$= e^{\frac{\pi}{2}}$ **[1 Mark]**

26. Evaluate the definite integral $\int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$ **[4 Marks]**

Solution:

Let $I = \int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\frac{(\sin x \cos x)}{\cos^4 x}}{\frac{(\cos^4 x + \sin^4 x)}{\cos^4 x}} dx$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\tan x \sec^2 x}{1 + \tan^4 x} dx$ **[2 Marks]**

Let $\tan^2 x = t \Rightarrow 2 \tan x \sec^2 x\, dx = dt$

When $x = 0, t = 0$, when $x = \frac{\pi}{4}, t = 1$

$\therefore I = \frac{1}{2}\int_0^1 \frac{dt}{1+t^2}$ **[1 Mark]**

$= \frac{1}{2}[\tan^{-1} t]_0^1$

$= \frac{1}{2}[\tan^{-1} 1 - \tan^{-1} 0]$

$= \frac{1}{2}\left[\frac{\pi}{4}\right]$

$= \frac{\pi}{8}$ **[1 Mark]**

27. Evaluate the definite integral $\int_0^{\frac{\pi}{2}} \frac{\cos^2 x\, dx}{\cos^2 x + 4 \sin^2 x}$ **[4 Marks]**

Solution:

Let $I=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\cos^2 x+4\sin^2 x}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\cos^2 x+4(1-\cos^2 x)}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\cos^2 x+4-4\cos^2 x}dx$

$\Rightarrow I=\frac{-1}{3}\int_0^{\frac{\pi}{2}}\frac{4-3\cos^2 x-4}{4-3\cos^2 x}dx$

$\Rightarrow I=\frac{-1}{3}\int_0^{\frac{\pi}{2}}\frac{4-3\cos^2 x}{4-3\cos^2 x}dx+\frac{1}{3}\int_0^{\frac{\pi}{2}}\frac{4}{4-3\cos^2 x}dx$

$\Rightarrow I=\frac{-1}{3}\int_0^{\frac{\pi}{2}}1\,dx+\frac{1}{3}\int_0^{\frac{\pi}{2}}\frac{4\sec^2 x}{4\sec^2 x-3}dx$

$\Rightarrow I=\frac{-1}{3}[x]_0^{\frac{\pi}{2}}+\frac{1}{3}\int_0^{\frac{\pi}{2}}\frac{4\sec^2 x}{4(1+\tan^2 x)-3}dx$

$\Rightarrow I=-\frac{\pi}{6}+\frac{2}{3}\int_0^{\frac{\pi}{2}}\frac{2\sec^2 x}{1+4\tan^2 x}dx$ ...(i) **[2 Marks]**

Now consider, $\int_0^{\frac{\pi}{2}}\frac{2\sec^2 x}{1+4\tan^2 x}dx$

Let $2\tan x=t\Rightarrow 2\sec^2 x\,dx=dt$

When $x=0, t=0$, when $x=\frac{\pi}{2}, t\to\infty$

$\Rightarrow \int_0^{\frac{\pi}{2}}\frac{2\sec^2 x}{1+4\tan^2 x}dx=\int_0^{\infty}\frac{dt}{1+t^2}$ **[1 Mark]**

$=[\tan^{-1}t]_0^{\infty}$

$=[\tan^{-1}(\infty)-\tan^{-1}(0)]$

$=\frac{\pi}{2}$

Hence, from (i), we get

$I=-\frac{\pi}{6}+\frac{2}{3}\left[\frac{\pi}{2}\right]=\frac{\pi}{3}-\frac{\pi}{6}=\frac{\pi}{6}$ **[1 Mark]**

28. Evaluate the definite integral $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sin x+\cos x}{\sqrt{\sin 2x}}dx$ **[4 Marks]**

Solution:

Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x+\cos x}{\sqrt{\sin 2x}} dx$

$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x+\cos x)}{\sqrt{-(-\sin 2x)}} dx$

$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x+\cos x}{\sqrt{-(-1+1-2\sin x\cos x)}} dx$

$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x+\cos x)}{\sqrt{1-(\sin^2 x+\cos^2 x-2\sin x\cos x)}} dx$

$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x+\cos x)dx}{\sqrt{1-(\sin x-\cos x)^2}}$ **[1 Mark]**

Let $(\sin x - \cos x) = t \Rightarrow (\sin x + \cos x)dx = dt$

When $x = \frac{\pi}{6}, t = \left(\frac{1-\sqrt{3}}{2}\right)$, when $x = \frac{\pi}{3}, t = \left(\frac{\sqrt{3}-1}{2}\right)$

$I = \int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$

$\Rightarrow I = \int_{-\left(\frac{\sqrt{3}-1}{2}\right)}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$ **[1 Mark]**

As $\frac{1}{\sqrt{1-(-t)^2}} = \frac{1}{\sqrt{1-t^2}}$, hence, $\frac{1}{\sqrt{1-t^2}}$ is an even function.

As we know that if $f(x)$ is an even function, then

$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

$\Rightarrow I = 2\int_0^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$

$= [2\sin^{-1} t]_0^{\frac{\sqrt{3}-1}{2}}$

$= 2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$ **[2 Marks]**

29. Evaluate the definite integral $\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$ **[4 Marks]**

Solution:

Let $I = \int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$

$I = \int_0^1 \frac{1}{(\sqrt{1+x}-\sqrt{x})} \times \frac{(\sqrt{1+x}+\sqrt{x})}{(\sqrt{1+x}+\sqrt{x})} dx$ **[1 Mark]**

$= \int_0^1 \frac{\sqrt{1+x}+\sqrt{x}}{1+x-x} dx$

$= \int_0^1 \sqrt{1+x}\, dx + \int_0^1 \sqrt{x}\, dx$ **[1 Mark]**

$= \left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^1 + \left[\frac{2}{3}(x)^{\frac{3}{2}}\right]_0^1$ **[1 Mark]**

$= \frac{2}{3}\left[(2)^{\frac{3}{2}} - 1\right] + \frac{2}{3}[1]$

$= \frac{2}{3}(2)^{\frac{3}{2}}$

$= \frac{2.2\sqrt{2}}{3}$

$= \frac{4\sqrt{2}}{3}$ **[1 Mark]**

30. Evaluate the definite integral $\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9+16\sin 2x} dx$ **[4 Marks]**

Solution:

Let $I = \int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{9+16\sin 2x} dx$

Again, let $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)dx = dt$

When $x = 0, t = -1$, when $x = \frac{\pi}{4}, t = 0$

$\Rightarrow (\sin x - \cos x)^2 = t^2$ **[1 Mark]**

$\Rightarrow \sin^2 x + \cos^2 x - 2\sin x \cos x = t^2$

$\Rightarrow 1 - \sin 2x = t^2$

$\Rightarrow \sin 2x = 1 - t^2$

$\therefore I = \int_{-1}^{0} \frac{dt}{9+16(1-t^2)}$ [1 Mark]

$= \int_{-1}^{0} \frac{dt}{9+16-16t^2}$

$= \int_{-1}^{0} \frac{dt}{25-16t^2} = \int_{-1}^{0} \frac{dt}{(5)^2-(4t)^2}$

$= \frac{1}{4}\left[\frac{1}{2(5)} \log\left|\frac{5+4t}{5-4t}\right|\right]_{-1}^{0}$

$= \frac{1}{40}\left[\log(1) - \log\left|\frac{1}{9}\right|\right]$

$= \frac{1}{40} \log 9$ [2 Marks]

31. Evaluate the definite integral $\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\, dx$ **[6 Marks]**

Solution:

Let $I = \int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\, dx = \int_0^{\frac{\pi}{2}} 2 \sin x \cos x \tan^{-1}(\sin x)\, dx$

Again, let $\sin x = t \Rightarrow \cos x\, dx = dt$

When $x = 0, t = 0$, when $x = \frac{\pi}{2}, t = 1$

$\Rightarrow I = 2\int_0^1 t \tan^{-1}(t)\, dt$ ...(i) [2 Marks]

Now, take $\int t \cdot \tan^{-1} t\, dt = \tan^{-1} t \cdot \int t\, dt - \int \left\{\frac{d}{dt}(\tan^{-1} t) \int t\, dt\right\} dt$

$= \tan^{-1} t \cdot \frac{t^2}{2} - \int \frac{1}{1+t^2} \cdot \frac{t^2}{2}\, dt$

$= \frac{t^2 \tan^{-1} t}{2} - \frac{1}{2}\int \frac{t^2+1-1}{1+t^2}\, dt$

$= \frac{t^2 \tan^{-1} t}{2} - \frac{1}{2}\int 1\, dt + \frac{1}{2}\int \frac{1}{1+t^2}\, dt$

$= \frac{t^2 \tan^{-1} t}{2} - \frac{1}{2} \cdot t + \frac{1}{2}\tan^{-1} t$

$\Rightarrow \int_0^1 t \cdot \tan^{-1} t\, dt = \left[\frac{t^2 \cdot \tan^{-1} t}{2} - \frac{t}{2} + \frac{1}{2}\tan^{-1} t\right]_0^1$ [2 Marks]

$= \frac{1}{2}\left[\frac{\pi}{4} - 1 + \frac{\pi}{4}\right]$

$= \frac{1}{2}\left[\frac{\pi}{2} - 1\right] = \frac{\pi}{4} - \frac{1}{2}$

From equation (i), we get

$I = 2\left[\frac{\pi}{4} - \frac{1}{2}\right] = \frac{\pi}{2} - 1$ **[2 Marks]**

32. Evaluate the definite integral $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$ **[6 Marks]**

Solution:

Let $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$ ...(i)

$I = \int_0^{\pi} \left\{\frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)}\right\} dx \quad \left(\because \int_0^a f(x)dx = \int_0^a f(a-x)\, dx\right)$

$\Rightarrow I = \int_0^{\pi} \left\{\frac{-(\pi - x)\tan x}{-(\sec x + \tan x)}\right\} dx$

$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \tan x} dx$ ...(ii) **[2 Marks]**

Adding (i) and (ii), we get

$2I = \int_0^{\pi} \frac{\pi \tan x}{\sec x + \tan x} dx$ **[1 Mark]**

$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \frac{\sin x}{\cos x}} dx$

$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\sin x + 1 - 1}{1 + \sin x} dx$

$\Rightarrow 2I = \pi \int_0^{\pi} 1 \cdot dx - \pi \int_0^{\pi} \frac{1}{1 + \sin x} dx$

$\Rightarrow 2I = \pi[x]_0^{\pi} - \pi \int_0^{\pi} \frac{1 - \sin x}{\cos^2 x} dx$

$\Rightarrow 2I = \pi^2 - \pi \int_0^{\pi} (\sec^2 x - \tan x \sec x) dx$ **[1 Mark]**

$\Rightarrow 2I = \pi^2 - \pi[\tan x - \sec x]_0^{\pi}$

$\Rightarrow 2I = \pi^2 - \pi[\tan \pi - \sec \pi - \tan 0 + \sec 0]$

$\Rightarrow 2I = \pi^2 - \pi[0 - (-1) - 0 + 1]$

$\Rightarrow 2I = \pi^2 - 2\pi$

$\Rightarrow 2I = \pi(\pi - 2)$

$\Rightarrow I = \frac{\pi}{2}(\pi - 2)$ **[2 Marks]**

33. Evaluate the definite integral $\int_1^4 [|x-1| + |x-2| + |x-3|]dx$ **[6 Marks]**

Solution:

Let $I = \int_1^4 [|x-1| + |x-2| + |x-3|]dx$

$\Rightarrow I = \int_1^4 |x-1|dx + \int_1^4 |x-2|dx + \int_1^4 |x-3|dx$

$I = I_1 + I_2 + I_3$ ...(i) **[1 Mark]**

where, $I_1 = \int_1^4 |x-1|dx, I_2 = \int_1^4 |x-2|dx$ and $I_3 = \int_1^4 |x-3|dx$ **[1 Mark]**

$I_1 = \int_1^4 |x-1|dx$

$(x-1) \geq 0$ for $1 \leq x \leq 4$

$\therefore I_1 = \int_1^4 (x-1)dx$

$\Rightarrow I_1 = \left[\frac{x^2}{2} - x\right]_1^4$

$\Rightarrow I_1 = \left[8 - 4 - \frac{1}{2} + 1\right] = \frac{9}{2}$ ...(ii) **[1 Mark]**

$I_2 = \int_1^4 |x-2|dx$

$x - 2 \geq 0$ for $2 \leq x \leq 4$ and $x - 2 \leq 0$ for $1 \leq x \leq 2$

$\therefore I_2 = \int_1^2 (2-x)dx + \int_2^4 (x-2)dx$

$\Rightarrow I_2 = \left[2x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^4$

$\Rightarrow I_2 = \left[4 - 2 - 2 + \frac{1}{2}\right] + [8 - 8 - 2 + 4]$

$\Rightarrow I_2 = \frac{1}{2} + 2 = \frac{5}{2}$ ...(iii) **[1 Mark]**

$I_3 = \int_1^4 |x-3|dx$

$x-3 \geq 0$ for $3 \leq x \leq 4$ and $x-3 \leq 0$ for $1 \leq x \leq 3$

$\therefore I_3 = \int_1^3 (3-x)dx + \int_3^4 (x-3)dx$

$\Rightarrow I_3 = \left[3x - \frac{x^2}{2}\right]_1^3 + \left[\frac{x^2}{2} - 3x\right]_3^4$

$\Rightarrow I_3 = \left[9 - \frac{9}{2} - 3 + \frac{1}{2}\right] + \left[8 - 12 - \frac{9}{2} + 9\right]$

$\Rightarrow I_3 = [6-4] + \left[\frac{1}{2}\right] = \frac{5}{2}$ ...(iv) **[1 Mark]**

From equations (i), (ii), (iii) and (iv), we get

$I = \frac{9}{2} + \frac{5}{2} + \frac{5}{2} = \frac{19}{2}$ **[1 Mark]**

34. Prove that $\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log\frac{2}{3}$ **[6 Marks]**

Solution:

Let $I = \int_1^3 \frac{dx}{x^2(x+1)}$

Again, let $\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$ **[1 Mark]**

$\Rightarrow I = Ax(x+1) + B(x+1) + C(x^2)$

$\Rightarrow I = Ax^2 + Ax + Bx + B + Cx^2$ **[1 Mark]**

After equating the coefficients of x^2, x and constant term, we get

$A + C = 0$

$A + B = 0$

$B = 1$

after solving these equations, we get

$A = -1, C = 1$ and $B = 1$ **[1 Mark]**

$\therefore \frac{1}{x^2(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)}$

$\Rightarrow I = \int_1^3 \left\{-\frac{1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)}\right\} dx$ **[1 Mark]**

$= \left[-\log x - \frac{1}{x} + \log(x+1)\right]_1^3$

$= \left[\log\left(\frac{x+1}{x}\right) - \frac{1}{x}\right]_1^3$

$= \log\left(\frac{4}{3}\right) - \frac{1}{3} - \log\left(\frac{2}{1}\right) + 1$

$= \log 4 - \log 3 - \log 2 + \frac{2}{3}$

$= \log 2 - \log 3 + \frac{2}{3}$

$= \log\left(\frac{2}{3}\right) + \frac{2}{3}$

Thus, its proved **[2 Marks]**

35. Prove that $\int_0^1 xe^x dx = 1$ **[2 Marks]**

Solution:

Consider $I = \int_0^1 xe^x\, dx$

Using integration by parts, we get

$I = x\int_0^1 e^x dx - \int_0^1 \left\{\left(\frac{d}{dx}(x)\right)\int e^x dx\right\} dx$ **[1 Mark]**

$= [xe^x]_0^1 - \int_0^1 e^x dx$

$= [xe^x]_0^1 - [e^x]_0^1$

$= e - e + 1$

$= 1$

Therefore, it's proved [1 Mark]

36. Prove that $\int_{-1}^{1} x^{17} \cos^4 x dx = 0$ [2 Marks]

Solution:

Let $I = \int_{-1}^{1} x^{17} \cos^4 x dx$

Again, let $f(x) = x^{17} \cos^4 x$

$\Rightarrow f(-x) = (-x)^{17} \cos^4(-x) = -x^{17} \cos^4 x = -f(x)$

Hence, $f(x)$ is an odd function. [1 Mark]

We know that if $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)dx = 0$

$\therefore I = \int_{-1}^{1} x^{17} \cos^4 x dx = 0$

Hence, it's proved. [1 Mark]

37. Prove that $\int_0^{\frac{\pi}{2}} \sin^3 x\, dx = \frac{2}{3}$ [4 Marks]

Solution:

Consider $I = \int_0^{\frac{\pi}{2}} \sin^3 x\, dx$

$I = \int_0^{\frac{\pi}{2}} \sin^2 x \cdot \sin x\, dx$

$= \int_0^{\frac{\pi}{2}} (1 - \cos^2 x) \sin x\, dx$

$= \int_0^{\frac{\pi}{2}} \sin x\, dx - \int_0^{\frac{\pi}{2}} \cos^2 x \cdot \sin x\, dx$ [2 Marks]

$= [-\cos x]_0^{\frac{\pi}{2}} + \left[\frac{\cos^3 x}{3}\right]_0^{\frac{\pi}{2}}$

$= 1 + \frac{1}{3}[-1] = 1 - \frac{1}{3} = \frac{2}{3}$

Thus, its proved **[2 Marks]**

38. Prove that $\int_0^{\frac{\pi}{4}} 2\tan^3 x dx = 1 - \log 2$ **[4 Marks]**

Solution:

Consider $I = \int_0^{\frac{\pi}{4}} 2\tan^3 x dx$

$I = 2\int_0^{\frac{\pi}{4}} \tan^2 x \tan x\, dx$

$$= 2\int_0^{\frac{\pi}{4}} (\sec^2 x - 1)\tan x\, dx$$

$= 2\int_0^{\frac{\pi}{4}} \sec^2 x \tan x\, dx - 2\int_0^{\frac{\pi}{4}} \tan x\, dx$ **[2 Marks]**

$$= 2\left[\frac{\tan^2 x}{2}\right]_0^{\frac{\pi}{4}} + 2[\log\cos x]_0^{\frac{\pi}{4}}$$

$$= 1 + 2\left[\log\cos\frac{\pi}{4} - \log\cos 0\right]$$

$$= 1 + 2\left[\log\frac{1}{\sqrt{2}} - \log 1\right]$$

$$= 1 - \log 2 - \log 1 = 1 - \log 2$$

Hence, it's proved. **[2 Marks]**

39. Prove that $\int_0^1 \sin^{-1} x dx = \frac{\pi}{2} - 1$ **[4 Marks]**

Solution:

Let's consider $I=\int_0^1 \sin^{-1} x dx$

$\Rightarrow I=\int_0^1 \sin^{-1} x \cdot 1 \cdot dx$ **[1 Mark]**

Using integration by parts, we get

$$I=[\sin^{-1} x \cdot x]_0^1-\int_0^1 \frac{1}{\sqrt{1-x^2}} \cdot x dx$$

$=[x \sin^{-1} x]_0^1+\frac{1}{2} \int_0^1 \frac{(-2x)}{\sqrt{1-x^2}} dx$ **[1 Mark]**

Again Let $1-x^2=t \Rightarrow -2x\, dx=dt$

When $x=0, t=1$, when $x=1, t=0$

$I=[x \sin^{-1} x]_0^1+\frac{1}{2} \int_1^0 \frac{dt}{\sqrt{t}}$ **[1 Mark]**

$=[x \sin^{-1} x]_0^1+\frac{1}{2}[2\sqrt{t}]_1^0$

$=\sin^{-1}(1)+[-\sqrt{1}]$

$=\frac{\pi}{2}-1$

Hence, it's proved. **[1 Mark]**

40. Evaluate $\int_0^1 e^{2-3x} dx$ as a limit of a sum. **[6 Marks]**

Solution:

Let's consider $I=\int_0^1 e^{2-3x} dx$

We know that,

$\int_a^b f(x) dx=(b-a) \lim_{n \to \infty} \frac{1}{n}[f(a)+f(a+h)+\cdots+f(a+(n-1)h)]$ **[1 Mark]**

Where, $h=\frac{b-a}{n}$

For, $a=0, b=1$ and $f(x)=e^{2-3x}$

$\Rightarrow h = \frac{1-0}{n} = \frac{1}{n}$ [1 Mark]

$\therefore \int_0^4 e^{2-3x} dx = (1-0) \lim_{n\to\infty} \frac{1}{n}[f(0) + f(0+h) + \cdots + f(0+(n-1)h)]$ [1 Mark]

$$= \lim_{n\to\infty} \frac{1}{n}\left[e^2 + e^{2-3h} + \cdots e^{2-3(n-1)h}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[e^2\left\{1 + e^{-3h} + e^{-6h} + e^{-9h} + \cdots e^{-3(n-1)h}\right\}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[e^2\left\{\frac{1-\left(e^{-3h}\right)^n}{1-\left(e^{-3h}\right)}\right\}\right]$$ [1 Mark]

$$= \lim_{n\to\infty} \frac{1}{n}\left[e^2\left\{\frac{1-e^{-\frac{3}{n}\times n}}{1-e^{-\frac{3}{n}}}\right\}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{e^2(1-e^{-3})}{1-e^{-\frac{3}{n}}}\right]$$

$$= e^2(e^{-3}-1) \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{e^{-\frac{3}{n}}-1}\right]$$

$$= e^2(e^{-3}-1) \lim_{n\to\infty} \left(-\frac{1}{3}\right)\left[\frac{-\frac{3}{n}}{e^{-\frac{3}{n}}-1}\right]$$

$$= \frac{-e^2(e^{-3}-1)}{3} \lim_{n\to\infty}\left[\frac{-\frac{3}{n}}{e^{-\frac{3}{n}}-1}\right]$$

$$= \frac{-e^2(e^{-3}-1)}{3}(1) \quad \left[\lim_{n\to\infty} \frac{x}{e^x-1}\right]$$

$$= \frac{-e^{-1}+e^2}{3}$$

$= \frac{1}{3}\left(e^2 - \frac{1}{e}\right)$ [2 Marks]

41. $\int \frac{dx}{e^x+e^{-x}}$ is equal to [2 Marks]

(A) $\tan^{-1}(e^x) + C$

(B) $\tan^{-1}(e^{-x})+C$

(C) $\log(e^{x}-e^{-x})+C$

(D) $\log(e^{x}+e^{-x})+C$

Solution:

(A)

Consider $I=\int\frac{dx}{e^{x}+e^{-x}}dx=\int\frac{e^{x}}{e^{2x}+1}dx$

Again, let $e^{x}=t\Rightarrow e^{x}dx=dt$

$\therefore I=\int\frac{dt}{1+t^{2}}$ **[1 Mark]**

$=\tan^{-1}t+C$

$=\tan^{-1}(e^{x})+C$

Therefore, (A) is the correct answer. **[1 Mark]**

42. $\int\frac{\cos 2x}{(\sin x+\cos x)^{2}}dx$ is equal to **[4 Marks]**

(A) $\frac{-1}{\sin x+\cos x}+C$

(B) $\log|\sin x+\cos x|+C$

(C) $\log|\sin x-\cos x|+C$

(D) $\frac{1}{(\sin x+\cos x)^{2}}$

Solution:

(B)

Consider $I=\frac{\cos 2x}{(\cos x+\sin x)^{2}}$

$$I=\int\frac{\cos^{2}x-\sin^{2}x}{(\cos x+\sin x)^{2}}dx$$

$$=\int\frac{(\cos x+\sin x)(\cos x-\sin x)}{(\cos x+\sin x)^{2}}dx$$

$= \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$ **[2 Marks]**

Also, let $\cos x + \sin x = t \Rightarrow (\cos x - \sin x)dx = dt$ **[1 Mark]**

$\therefore I = \int \frac{dt}{t}$

$= \log|t| + C$

$= \log|\cos x + \sin x| + C$

Hence, (B) is the correct answer. **[1 Mark]**

43. If $f(a+b-x) = f(x)$, then $\int_a^b x\, f(x)\, dx$ is equal to **[4 Marks]**

(A) $\frac{a+b}{2} \int_a^b f(b-x)\, dx$

(B) $\frac{a+b}{2} \int_a^b f(b+x) dx$

(C) $\frac{b-a}{2} \int_a^b f(x) dx$

(D) $\frac{a+b}{2} \int_a^b f(x)\, dx$

Solution:

(D)

Consider $I = \int_a^b x\, f(x)\, dx$ ...(i)

$I = \int_a^b (a+b-x) f(a+b-x) dx \quad \left(\because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx\right)$ **[1 Mark]**

$$\Rightarrow I = \int_a^b (a+b-x) f(x) dx$$

$\Rightarrow I = (a+b) \int_a^b f(x) dx - I$ [Using (i)] **[1 Mark]**

$$\Rightarrow I + I = (a+b) \int_a^b f(x) dx$$

$$\Rightarrow 2I = (a+b)\int_a^b f(x)dx$$

$$\Rightarrow I = \left(\frac{a+b}{2}\right)\int_a^b f(x)dx$$

Hence, (D) is the correct answer. **[2 Marks]**

44. The value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$ is **[4 Marks]**

(A) 1

(B) 0

(C) -1

(D) $\frac{\pi}{4}$

Solution:

(B)

Consider $I = \int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$

$\Rightarrow I = \int_0^1 \tan^{-1}\left(\frac{x-(1-x)}{1+x(1-x)}\right)dx$

$\Rightarrow I = \int_0^1 [\tan^{-1}x - \tan^{-1}(1-x)]dx$ ......(i) **[1 Mark]**

$\Rightarrow I = \int_0^1 [\tan^{-1}(1-x) - \tan^{-1}(1-1+x)]dx$

$\Rightarrow I = \int_0^1 [\tan^{-1}(1-x) - \tan^{-1}(x)]dx$

$\Rightarrow I = \int_0^1 [\tan^{-1}(1-x) - \tan^{-1}(x)]dx$ ...(ii) **[1 Mark]**

Adding (i) and (ii), we get,

$2I = \int_0^1 (\tan^{-1}x - \tan^{-1}(1-x) + \tan^{-1}(1-x) - \tan^{-1}x)dx$

$\Rightarrow 2I = 0$

$\Rightarrow I = 0$

Hence, (B) is the correct answer. **[2 Marks]**