

CBSE NCERT Solutions for Class 12 Maths Chapter 06

Back of Chapter Questions

Exercise 6.1

1. Find the rate of change of the area of a circle with respect to its radius r when
[2 Marks]

(a) $r = 3$ cm

(b) $r = 4$ cm

Solution:

Step 1:

We know that:

The area of a circle (A) with radius (r) is given by,

$$A = \pi r^2$$

Step 2:

Now, the rate of change of the area with respect to its radius is given by,

$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$$

$$\frac{dA}{dr} = 2\pi r$$

[1 Mark]

Step 3:

(a). **Given:** $r = 3$ cm,

$$\frac{dA}{dr} = 2\pi(3) = 6\pi$$

$[\frac{1}{2}$ Mark]

Therefore, the area of the circle is changing at the rate of 6π cm²/s when its radius is 3 cm.

Step 4:

(b). **Given:** $r = 4$ cm,

$$\frac{dA}{dr} = 2\pi(4) = 8\pi$$

$[\frac{1}{2}$ Mark]

Hence, the area of the circle is changing at the rate of 8π cm²/s when its radius is 4 cm.

2. The volume of a cube is increasing at the rate of $8 \text{ cm}^3/\text{s}$. How fast is the surface area increasing when the length of an edge is 12 cm ? [4 Mark]

Solution:

Step 1:

Given:

The volume of a cube is increasing at the rate of $8 \text{ cm}^3/\text{s}$.

$$\frac{dV}{dt} = 8 \text{ cm}^3/\text{s}$$

Step 2:

So, let x be the length of a side, V be the volume, and S be the surface area of the cube.

Then, $V = x^3$ and $S = 6x^2$ where x is a function of time t .

Step 3:

$$\frac{dV}{dt} = 8 \text{ cm}^3/\text{s}$$

Then, by using the chain rule, we have:

$$8 = \frac{dV}{dt} = \frac{d}{dt}(x^3) = \frac{d}{dx}(x^3) \cdot \frac{dx}{dt} = 3x^2 \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{8}{3x^2} \quad (1) \quad [1\frac{1}{2}\text{Mark}]$$

Step 4:

Now,

$$S = 6x^2$$

Differentiate with respect to t

$$\frac{dS}{dt} = \frac{d}{dt}(6x^2) = \frac{d}{dx}(6x^2) \cdot \frac{dx}{dt} \quad [\text{By chain rule}]$$

$$= 12x \cdot \frac{dx}{dt} = 12x \cdot \left(\frac{8}{3x^2}\right) = \frac{32}{x} \quad [1\frac{1}{2}\text{Mark}]$$

Step 5:

Therefore, when $x = 12 \text{ cm}$,

$$\frac{dS}{dt} = \frac{32}{12} \text{ cm}^2/\text{s} = \frac{8}{3} \text{ cm}^2/\text{s}. \quad [1 \text{ Mark}]$$

Hence, if the length of the edge of the cube is 12 cm , then the surface area is increasing at the rate of $\frac{8}{3} \text{ cm}^2/\text{s}$.

3. The radius of a circle is increasing uniformly at the rate of 3 cm/s. Find the rate at which the area of the circle is increasing when the radius is 10 cm. [2 Marks]

Solution:

Step 1:

Given:

The radius of a circle is increasing uniformly at the rate of 3 cm/s.

$$\frac{dr}{dt} = 3 \text{ cm/s}$$

Step 2:

The area of a circle (A) with radius (r) is given by,

$$A = \pi r^2$$

Step 3:

Now, the rate of change of area (A) with respect to time (t) is given by,

$$\frac{dA}{dt} = \frac{d}{dt}(\pi r^2) \cdot \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \quad [\text{By chain rule}] \quad [1 \text{ Mark}]$$

Step 4:

It is given that,

$$\frac{dr}{dt} = 3 \text{ cm/s}$$

$$\therefore \frac{dA}{dt} = 2\pi r(3) = 6\pi r \quad [\frac{1}{2} \text{ Mark}]$$

Step 5:

Thus, when $r = 10$ cm,

$$\frac{dA}{dt} = 6\pi(10) = 60\pi \text{ cm}^2/\text{s} \quad [\frac{1}{2} \text{ Mark}]$$

Hence, the rate at which the area of the circle is increasing when the radius is 10 cm is $60\pi \text{ cm}^2/\text{s}$.

4. An edge of a variable cube is increasing at the rate of 3 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long? [2 Marks]

Solution:

Step 1:

Given:

An edge of a variable cube is increasing at the rate of 3 cm/s

$$\frac{dx}{dt} = 3 \text{ cm/s}$$

Step 2:

Let x be the length of a side and V be the volume of the cube. Then,

$$V = x^3$$

By differentiating, we get

$$\frac{dV}{dt} = 3x^2 \cdot \frac{dx}{dt} \text{ (By chain rule) } \quad \left[\frac{1}{2}\text{Mark}\right]$$

Step 3:

It is given that,

$$\frac{dx}{dt} = 3 \text{ cm/s}$$

$$\frac{dV}{dt} = 3x^2(3) = 9x^2 \quad \left[\frac{1}{2}\text{Mark}\right]$$

Step 4:

Thus, when $x = 10 \text{ cm}$,

$$\frac{dV}{dt} = 9(10)^2 = 900 \text{ cm}^3/\text{s} \quad [1 \text{ Mark}]$$

Therefore, the volume of the cube is increasing at the rate of $900 \text{ cm}^3/\text{s}$ when the edge is 10 cm long.

5. A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing?

[1 Mark]

Solution:**Step 1:****Given:**

The rate of change of radius = 5 cm/s.

Step 2:

The area of a circle (A) with radius (r) is given by $A = \pi r^2$

Therefore, the rate of change of area (A) with respect to time (t) is given by,

[By chain rule]

$$\frac{dA}{dt} = \frac{d}{dt} (\pi r^2) = \frac{d}{dr} (\pi r^2) \frac{dr}{dt} = 2\pi r \frac{dr}{dt} \quad [1 \text{ Mark}]$$

Step 3:

It is given that $\frac{dr}{dt} = 5 \text{ cm/s}$

Hence, when $r = 8 \text{ cm}$,

$$\frac{dA}{dt} = 2\pi(8)(5) = 80\pi r \quad [1 \text{ Mark}]$$

Therefore, when the radius of the circular wave is 8 cm, the enclosed area is increasing at the rate of $80\pi \text{ cm}^2/\text{s}$.

6. The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference? [2 Marks]

Solution:

Step 1:

Given:

The rate of change of radius = 0.7 cm/s

Step 2:

The circumference of a circle (C) with radius (r) is given by $C = 2\pi r$.

Therefore, the rate of change of circumference (C) with respect to time (t) is given by,

$$\frac{dC}{dt} = \frac{dC}{dr} \cdot \frac{dr}{dt} \quad (\text{By chain rule})$$

$$\frac{d}{dr} (2\pi r) \frac{dr}{dt}$$

$$\frac{dC}{dt} = 2\pi \frac{dr}{dt} \quad [1\frac{1}{2} \text{ Mark}]$$

Step 3:

It is given that $\frac{dr}{dt} = 0.7 \text{ cm/s}$

$$\frac{dC}{dt} = 2\pi \times 0.7$$

$$\frac{dC}{dt} = 1.4\pi \quad [\frac{1}{2} \text{ Mark}]$$

7. The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When $x = 8$ cm and $y = 6$ cm, find the rates of change of

(a) the perimeter and [1 Mark]

(b) the area of the rectangle [1 mark]

Solution:

Step 1:

Given:

The length (x) is decreasing at the rate of 5 cm/minute and the width (y) is increasing at the rate of 4 cm/minute.

Step 2:

We have:

$$\frac{dx}{dt} = -5 \text{ cm/min and } \frac{dy}{dt} = 4 \text{ cm/min}$$

Step 3:

(a) The perimeter (P) of a rectangle is given by,

$$P = 2(x + y)$$

Differentiate with respect to time

$$\frac{dP}{dt} = \frac{d(2(x + y))}{dt}$$

$$\frac{dP}{dt} = 2 \frac{d(x + y)}{dt}$$

$$\therefore \frac{dP}{dt} = 2 \left(\frac{dx}{dt} + \frac{dy}{dt} \right) \left[\frac{1}{2} \text{Mark} \right]$$

Step 4:

$$= 2(-5 + 4) = -2 \text{ cm/min}$$

Hence, the perimeter is decreasing at the rate of 2 cm/min. $\left[\frac{1}{2} \text{Mark} \right]$

Step 5:

(b) The area (A) of a rectangle is given by,

$$A = x \times y$$

Differentiate with respect to time

$$\frac{dA}{dt} = \frac{d(x \cdot y)}{dt}$$

$$\frac{dA}{dt} = \frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt}$$

$$\frac{dA}{dt} = -5 \cdot y + 4 \cdot x$$

$$= -5y + 4x \quad \left[\frac{1}{2}\text{Mark}\right]$$

Step 6:

When $x = 8 \text{ cm}$ and $y = 6 \text{ cm}$,

$$\frac{dA}{dt} = (-5 \times 6 + 4 \times 8) \text{ cm}^2/\text{min}$$

$$\frac{dA}{dt} = 32 - 30$$

$$\frac{dA}{dt} = 2 \text{ cm}^2/\text{min} \quad \left[\frac{1}{2}\text{Mark}\right]$$

Therefore, the area of the rectangle is increasing at the rate of $2 \text{ cm}^2/\text{min}$.

8. A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm. [2 Mark]

Solution:

Step 1:

Given:

The balloon is inflated by pumping in 900 cubic centimeters of gas per second.

$$\frac{dV}{dt} = 900 \text{ cm}^3/\text{s}$$

Step 2:

The volume of a sphere (V) with radius (r) is given by,

$$V = \frac{4}{3}\pi r^3$$

Step 3:

\therefore Rate of change of volume (V) with respect to time (t) is given by,

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} \quad [\text{By chain rule}]$$

$$\frac{dV}{dt} = \frac{d}{dr} \left(\frac{4}{3}\pi r^3 \right) \cdot \frac{dr}{dt}$$

$$\frac{dv}{dt} = \frac{4}{3}\pi \frac{d(r^3)}{dt} \quad \left[\frac{1}{2}\text{Mark}\right]$$

Step 4:

It is given that $\frac{dV}{dt} = 900 \text{ cm}^3/\text{s}$

$$900 = \frac{4}{3}\pi \cdot \frac{d(r^3)}{dt} \times \frac{dr}{dt}$$

$$900 = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\therefore 900 = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{900}{4\pi r^2} = \frac{225}{\pi r^2} \quad [1 \text{ Mark}]$$

Step 5:

Therefore, when radius = 15 cm,

$$\frac{dr}{dt} = \frac{225}{\pi(15)^2} = \frac{1}{\pi} \quad [\frac{1}{2} \text{ Mark}]$$

Therefore, the rate at which the radius of the balloon increases when the radius is 15 cm is $\frac{1}{\pi} \text{ cm/s}$

9. A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the later is 10 cm. [2 Marks]

Solution:**Step 1:**

Given: Radius = 10 cm.

Step 2:

The volume of a sphere (V) with radius (r) is given by $V = \frac{4}{3}\pi r^3$.

Rate of change of volume (V) with respect to its radius (r) is given by,

$$\frac{dV}{dr} = \frac{d}{dr} \left(\frac{4}{3}\pi r^3 \right) = \frac{4}{3}\pi(3r^2) = 4\pi r^2 \quad [1 \text{ Mark}]$$

Step 3:

Therefore, when radius = 10 cm,

$$\frac{dV}{dr} = 4\pi(10)^2 = 400\pi \quad [1 \text{ Mark}]$$

Therefore, the volume of the balloon is increasing at the rate of $400\pi \text{ cm}^3/\text{s}$.

10. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall? [4 Marks]

Solution:

Step 1:

Given:

Length of the ladder = 5 m

$$\frac{dx}{dt} = 2 \text{ cm/s}$$

Step 2:

Let the height of the wall at which the ladder touches be = y m .

Let the foot of the ladder away from the wall be = x m.

Then, from Pythagoras theorem, we have: $x^2 + y^2 = 25$

$$\Rightarrow y = \sqrt{25 - x^2} \quad [1 \text{ Mark}]$$

Step 3:

So, the rate of change of height (y) with respect to time (t) is given by,

$$\frac{dy}{dt} = \frac{-x}{\sqrt{25-x^2}} \cdot \frac{dx}{dt} \quad [1 \text{ Mark}]$$

Step 4:

It is given that $\frac{dx}{dt} = 2 \text{ cm/s}$

$$\therefore \frac{dy}{dt} = \frac{-2x}{\sqrt{25-x^2}} \quad [1 \text{ Mark}]$$

Step 5:

Now, when $x = 4$ m, we have:

$$\frac{dy}{dt} = \frac{-2 \times 4}{\sqrt{25-4^2}} = -\frac{8}{3} \quad [1 \text{ Mark}]$$

Therefore, the height of the ladder on the wall is decreasing at the rate of $\frac{8}{3}$ cm/s

11. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y -coordinate is changing 8 times as fast as the x -coordinate.

Solution:

Step 1:

Given:

The equation of the curve is:

$$6y = x^3 + 2$$

Step 2:

The rate of change of the position of the particle with respect to time (t) is given by,

$$6 \frac{dy}{dt} = 3x^2 \frac{dx}{dt} + 0$$

$$\Rightarrow 2 \frac{dy}{dt} = x^2 \frac{dx}{dt} \quad [1 \text{ Mark}]$$

Step 3:

When the y -coordinate of the particle changes 8 times as fast as the

$$\left(\frac{dy}{dt} = 8 \frac{dx}{dt}\right) \text{ } x\text{-coordinate i.e.,}$$

$$2 \left(8 \frac{dx}{dt}\right) = x^2 \frac{dx}{dt}$$

$$\Rightarrow 16 \frac{dx}{dt} = x^2 \frac{dx}{dt}$$

$$\Rightarrow (x^2 - 16) \frac{dx}{dt} = 0$$

$$\Rightarrow x^2 = 16$$

$$\Rightarrow x = \pm 4 \quad [1 \text{ Mark}]$$

Step 4:

Now by substituting the value of x we get:

When $x = 4$,

$$y = \frac{4^3 + 2}{6} = \frac{66}{6} = 11 \quad [1 \text{ Mark}]$$

Step 5:

When $x = -4$,

$$y = \frac{(-4)^3 + 2}{6} = -\frac{62}{6} = -\frac{31}{3} \quad [1 \text{ Mark}]$$

Therefore, the points required on the curve are $(4, 11)$ and $\left(-4, -\frac{31}{3}\right)$.

12. The radius of an air bubble is increasing at the rate of $\frac{1}{2}$ cm/s. At what rate is the volume of the bubble increasing when the radius is 1 cm? [3 Marks]

Solution:**Step 1:****Given:**

$$\frac{dr}{dt} = \frac{1}{2} \text{ cm/s}$$

The air bubble is in the shape of a sphere.

Step 2:

Now, the volume of an air bubble (V) with radius (r) is given by,

$$V = \frac{4}{3}\pi r^3$$

Step 3:

The rate of change of volume (V) with respect to time (t) is given by,

$$\frac{dV}{dt} = \frac{4}{3}\pi \frac{d}{dr}(r^3) \cdot \frac{dr}{dt} \quad [\text{By chain rule}]$$

$$= \frac{4}{3}\pi(3r^2) \frac{dr}{dt}$$

$$= 4\pi r^2 \frac{dr}{dt} \quad [1 \frac{1}{2} \text{Mark}]$$

Step 4:

It is given that $\frac{dr}{dt} = \frac{1}{2} \text{ cm/s}$

Hence, when $r = 1 \text{ cm}$,

$$\frac{dV}{dt} = 4\pi(1)^2 \left(\frac{1}{2}\right) = 2\pi \text{ cm}^3/\text{s} \quad [1 \frac{1}{2} \text{Mark}]$$

Therefore, the rate at which the volume of the bubble is increasing at $2\pi \text{ cm}^3/\text{s}$.

13. A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x + 1)$ Find the rate of change of its volume with respect to x . [4 Marks]

Solution:**Step 1:****Given:**

$$\text{Diameter} = \frac{3}{2}(2x + 1)$$

The volume of a sphere (V) with radius (r) is given by,

$$V = \frac{4}{3}\pi r^3 \quad [\frac{1}{2}\text{Mark}]$$

Step 2:

It is given that:

$$\text{Diameter} = \frac{3}{2}(2x + 1)$$

$$\Rightarrow r = \frac{3}{4}(2x + 1) \quad [\frac{1}{2}\text{Mark}]$$

Step 3:

$$\therefore V = \frac{4}{3}\pi \left(\frac{3}{4}\right)^3 (2x + 1)^3$$

$$V = \frac{9}{16}\pi(2x + 1)^3 \quad [1 \text{ Mark}]$$

Step 4:

Hence, the rate of change of volume with respect to x is $\frac{dV}{dx}$

$$\therefore \frac{dV}{dx} = \frac{9}{16}\pi \frac{d}{dx}(2x + 1)^3$$

$$= \frac{9}{16}\pi \times 3(2x + 1)^2 \times 2 \quad [1 \text{ mark}]$$

Step 5:

$$= \frac{27}{8}\pi(2x + 1)^2 \quad [1 \text{ Mark}]$$

Hence, the rate of change of its volume with respect to x is $\frac{27}{8}\pi(2x + 1)^2$

14. Sand is pouring from a pipe at the rate of $12 \text{ cm}^3/\text{s}$. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm? [4 Marks]

Solution:

Step 1:

Given:

$$\frac{dV}{dt} = 12 \text{ cm}^3/\text{s}$$

$$h = \frac{1}{6}r$$

Step 2:

The volume of a cone (V) with radius (r) and height (h) is given by,

$$V = \frac{1}{3}\pi r^2 h$$

It is given that,

$$h = \frac{1}{6}r$$

$$\Rightarrow r = 6h$$

$$\therefore V = \frac{1}{3}\pi(6h)^2 h = 12\pi h^3 \quad \text{[1 Mark]}$$

Step 3;

Hence, the rate of change of volume with respect to time (t) is given by,

$$\frac{dV}{dt} = 12\pi \frac{d}{dh}(h^3) \cdot \frac{dh}{dt} \quad \text{[By chain rule]}$$

$$= 12\pi(3h^2) \frac{dh}{dt}$$

$$= 36\pi h^2 \frac{dh}{dt} \quad \text{[1 Mark]}$$

Step 4:

It is also given that $\frac{dV}{dt} = 12 \text{ cm}^3/\text{s}$

Therefore, when $h = 4 \text{ cm}$,

By substituting we get:

$$12 = 36\pi(4)^2 \frac{dh}{dt} \quad \text{[1 Mark]}$$

Step 5:

$$\Rightarrow \frac{dh}{dt} = \frac{12}{36\pi(16)} = \frac{1}{48\pi} \quad \text{[1 Mark]}$$

Therefore, when the height of the sand cone is 4 cm , its height is increasing at the rate of $\frac{1}{48\pi} \text{ cm/s}$.

15. The total cost $C(x)$ in Rupees associated with the production of x units of an item is given by

$$C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000 \quad \text{[4 Marks]}$$

Find the marginal cost when 17 units are produced.

Solution:

Step 1:

Given:

$$C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$$

Step 2:

Marginal cost is the rate of change of total cost with respect to output.

$$MC = \frac{d(c(x))}{dx} \quad \left[\frac{1}{2}\text{Mark}\right]$$

Step 2:

$$\text{Hence, Marginal cost (MC)} = \frac{dC}{dx} = 0.007(3x^2) - 0.003(2x) + 15$$

$$MC = 0.021x^2 - 0.006x + 1 \quad \left[1\frac{1}{2}\text{Mark}\right]$$

Step 3:

When $x = 17$, by substituting we get,

$$MC = 0.021(17^2) - 0.006(17) + 15$$

$$= 0.021(289) - 0.006(17) + 15$$

$$= 6.069 - 0.102 + 15$$

$$= 20.967 \quad [2 \text{ Marks}]$$

Therefore, when 17 units are produced, the marginal cost is ₹ 20.967.

16. The total revenue in Rupees received from the sale of x units of a product is given by

$$R(x) = 13x^2 + 26x + 15$$

Find the marginal revenue when $x = 7$. [2 marks]

Solution:**Step 1:****Given:**

$$R(x) = 13x^2 + 26x + 15$$

Step 2:

Marginal revenue is the rate of change of total revenue with respect to the number of units sold.

$$\text{Hence, the marginal Revenue (MR)} = \frac{dR}{dx} = 13(2x) + 26$$

$$\frac{dR}{dx} = 26x + 26 \quad [1 \text{ Mark}]$$

Step 3:

When $x = 7$, by substituting we get,

$$\begin{aligned} MR &= 26(7) + 26 \\ &= 182 + 26 = 208 \quad \text{[1 Mark]} \end{aligned}$$

Therefore, the required marginal revenue is ₹ 208.

17. The rate of change of the area of a circle with respect to its radius r at $r = 6$ cm is [2 marks]

- (A) 10π
- (B) 12π
- (C) 8π
- (D) 11π

Solution:**Step 1:****Given:**

$$r = 6 \text{ cm}$$

Step 2:

The area of a circle (A) with radius (r) is given by,

$$A = \pi r^2$$

Hence, the rate of change of the area with respect to its radius r is

$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2)$$

$$\frac{dA}{dr} = 2\pi r \quad \text{[1 Mark]}$$

Step 3:

When $r = 6$ cm,

$$\frac{dA}{dr} = 2\pi \times 6 = 12\pi \text{ cm}^2/\text{s} \quad \text{[1 Mark]}$$

Therefore, the required rate of change of the area of a circle is $12\pi \text{ cm}^2/\text{s}$.

Hence, the correct answer is B.

18. The total revenue in Rupees received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. The marginal revenue, when $x = 15$ is [2 marks]

- (A) 116
 (B) 96
 (C) 90
 (D) 126

Solution:

Step 1;

Given:

$$R(x) = 3x^2 + 36x + 5$$

Step 2:

Marginal revenue is the rate of change of total revenue with respect to the number of units sold.

$$\text{Hence, the Marginal Revenue (MR)} = \frac{dR}{dx} = 3(2x) + 36$$

$$(\text{MR}) = 6x + 36 \quad \text{[1 Mark]}$$

Step 3:

\therefore When $x = 15$,

$$\text{MR} = 6(15) + 36 = 90 + 36 = 126$$

Therefore, the required marginal revenue is ₹ 126.

Hence, the correct answer is D. [1 Mark]

Exercise 6.3

1. Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x = 4$. [2 Marks]

Solution:

Step 1:

The given curve is $y = 3x^4 - 4x$.

Step 2:

So, the slope of the tangent to the given curve at $x = 4$ is given by,

$$\left. \frac{dy}{dx} \right|_{x=4} = 12x^3 - 4 \Big|_{x=4} \quad \text{[1 Mark]}$$

Step 3:

$$\begin{aligned}
 &= 12(4)^2 - 4 \\
 &= 12(64) - 4 \\
 &= 768 - 4 \\
 &= 764 \text{ [1 Mark]}
 \end{aligned}$$

Hence, the required slope of the tangent is 764.

2. Find the slope of the tangent to the curve, $y = \frac{x-1}{x-2}$ at $x = 10$. [2 Marks]

Solution:

Step 1:

The given curve is $y = \frac{x-1}{x-2}$

Step 2:

$$\frac{dy}{dx} = \frac{d\left(\frac{x-1}{x-2}\right)}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{(x-2)(1) - (x-1)(1)}{(x-2)^2}$$

[By using quotient rule]

$$= \frac{x-2-x+1}{(x-2)^2}$$

$$\frac{dy}{dx} = \frac{-1}{(x-2)^2} \text{ [1 Mark]}$$

Step 3:

Hence, the slope of the tangent at $x = 10$ is given by,

$$\left. \frac{dy}{dx} \right|_{x=10} = \left. \frac{-1}{(x-2)^2} \right|_{x=10}$$

$$= \frac{-1}{(10-2)^2} = \frac{-1}{64} \text{ [1 Mark]}$$

Therefore, the slope of the tangent at $x = 10$ is $\frac{-1}{64}$.

3. Find the slope of the tangent to curve $y = x^3 - x + 1$ at the point whose x -coordinate is 2. [2 Mark]

Solution:

Step 1:

The given curve is $y = x^3 - x + 1$

Step 2:

By differentiating the given equation, we get

$$\frac{dy}{dx} = 3x^2 - 1$$

The slope of the tangent to a curve at (x_0, y_0) is $\left. \frac{dy}{dx} \right|_{(x_0, y_0)}$

It is given that $x_0 = 2$. **[1 Mark]**

Step 3:

Therefore, the slope of the tangent at the point where the x -coordinate is 2 is given by,

$$\begin{aligned} \left. \frac{dy}{dx} \right|_{x=2} &= 3x^2 - 1 \Big|_{x=2} \\ &= 3(2)^2 - 1 \\ &= 12 - 1 = 11 \quad \mathbf{[1 \text{ Mark}]} \end{aligned}$$

Hence, the slope of the tangent is 11.

4. Find the slope of the tangent to the curve $y = x^3 - 3x + 2$ at the point whose x -coordinate is 3.

[2 Marks]

Solution:

Step 1:

The given curve is $y = x^3 - 3x + 2$

Step 2:

On differentiating the given equation, we get

$$\frac{dy}{dx} = 3x^2 - 3$$

The slope of the tangent to a curve at (x_0, y_0) is $\left. \frac{dy}{dx} \right|_{(x_0, y_0)}$ **[1 Mark]**

Step 3:

Therefore, the slope of the tangent at the point where the x -coordinate is 3 is given by,

$$\begin{aligned} \left. \frac{dy}{dx} \right|_{x=3} &= 3x^2 - 3 \Big|_{x=3} \\ &= 3(3)^2 - 3 \end{aligned}$$

$$= 27 - 3 = 24 \text{ [1 Mark]}$$

Hence, the slope of the tangent is 24.

5. Find the slope of the normal to the curve $x = a\cos^3 \theta$, $y = a\sin^3 \theta$ at $\theta = \frac{\pi}{4}$ [4 Marks]

Solution:

Step 1:

It is given that $x = a\cos^3 \theta$ and $y = a\sin^3 \theta$.

Step 2:

On differentiating both the equations with respect to θ , we get

$$x = a\cos^3 \theta$$

$$\frac{dx}{d\theta} = 3a\cos^2 \theta(-\sin \theta)$$

$$\therefore \frac{dx}{d\theta} = -3a\cos^2 \theta \sin \theta \text{ [1 Mark]}$$

Step 3:

$$y = a\sin^3 \theta$$

$$\therefore \frac{dy}{d\theta} = 3a \sin^2 \theta (\cos \theta)$$

Step 3:

$$\text{Hence, } \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{3a\sin^2 \theta \cos \theta}{-3a\cos^2 \theta \sin \theta}$$

$$\frac{dy}{dx} = -\frac{\sin \theta}{\cos \theta} = -\tan \theta \text{ [1Mark]}$$

Step 4:

So, the slope of the tangent at $\theta = \frac{\pi}{4}$ is given by,

$$\left. \frac{dy}{dx} \right|_{\theta=\frac{\pi}{4}} = -\tan \theta \Big|_{\theta=\frac{\pi}{4}}$$

$$= -\tan \frac{\pi}{4} = -1 \text{ [1 Mark]}$$

Step 5:

Therefore, the slope of the normal at $\theta = \frac{\pi}{4}$ is given by,

$$\frac{1}{\text{slope of the tangent at } \theta} = \frac{-1}{-1} = 1 \text{ [1 Mark]}$$

Hence, the slope of the normal to the given curves is 1.

6. Find the slope of the normal to the curve $x = 1 - a \sin \theta, y = b \cos^2 \theta$ at $\theta = \frac{\pi}{2}$. [4 Marks]

Solution:

Step 1:

It is given that $x = 1 - a \sin \theta$ and $y = b \cos^2 \theta$.

Step 2:

By differentiating both the equations with respect to θ , we get

$$x = 1 - a \sin \theta$$

$$\therefore \frac{dx}{d\theta} = -a \cos \theta \quad \text{[1 Mark]}$$

Step 3:

$$y = b \cos^2 \theta$$

$$\therefore \frac{dy}{d\theta} = 2b \cos \theta (-\sin \theta) = -2b \sin \theta \cos \theta \quad \text{[1 Mark]}$$

Step 4:

Now,

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-2b \sin \theta \cos \theta}{-a \cos \theta} = \frac{2b}{a} \sin \theta$$

Hence, the slope of the tangent at $\theta = \frac{\pi}{2}$ is given by,

$$\begin{aligned} \left. \frac{dy}{dx} \right|_{\theta=\frac{\pi}{2}} &= \left. \frac{2b}{a} \sin \theta \right|_{\theta=\frac{\pi}{2}} \\ &= \frac{2b}{a} \sin \frac{\pi}{2} = \frac{2b}{a} \quad \text{[1 Mark]} \end{aligned}$$

Step 5:

Hence, the slope of the normal at $\theta = \frac{\pi}{2}$ is given by,

$$\frac{1}{\text{slope of the tangent at } \theta} = \frac{-1}{\left(\frac{2b}{a}\right)} = -\frac{a}{2b} \quad \text{[1 Mark]}$$

Therefore, the slope of the normal to the given curves is $-\frac{a}{2b}$.

7. Find points at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to the x -axis. [4 Marks]

Solution:**Step 1:**

The equation of the given curve is $y = x^3 - 3x^2 - 9x + 7$.

Step 2:

By differentiating with respect to x , we get

$$\frac{dy}{dx} = 3x^2 - 6x - 9 \quad \text{[1 Mark]}$$

Step 3:

Now, the tangent is parallel to the x -axis if the slope of the tangent is zero.

$$\therefore 3x^2 - 6x - 9 = 0$$

$$\Rightarrow x^2 - 2x - 3 = 0$$

$$(x - 3)(x + 1) = 0$$

$$x = 3 \text{ or } x = -1 \quad \text{[1 Mark]}$$

Step 4:

When $x = 3$,

$$\begin{aligned} y &= (3)^3 - 3(3)^2 - 9(3) + 7 \\ &= 27 - 27 - 27 + 7 = -20. \quad \text{[1 Mark]} \end{aligned}$$

Step 5:

When $x = -1$,

$$\begin{aligned} y &= (-1)^3 - 3(-1)^2 - 9(-1) + 7 \\ &= -1 - 3 + 9 + 7 = 12. \quad \text{[1 Mark]} \end{aligned}$$

Therefore, the points at which the tangent is parallel to the x -axis are $(3, -20)$ and $(-1, 12)$.

8. Find a point on the curve $y = (x - 2)^2$ at which the tangent is parallel to the chord joining the points $(2, 0)$ and $(4, 4)$. [4 Marks]

Solution:**Step 1:**

$$y = (x - 2)^2$$

Step 2:

If a tangent is parallel to the chord joining the points $(2, 0)$ and $(4, 4)$, then the slope of the tangent = the slope of the chord.

$$\therefore \text{The slope of the chord is } \frac{4-0}{4-2} = \frac{4}{2} = 2. \quad \text{[1 Mark]}$$

Step 3:

Now,

the slope of the tangent to the given curve at a point (x, y) is given by,

$$y = (x - 2)^2$$

On differentiating with respect to x , we get

$$\frac{dy}{dx} = 2(x - 2) \text{ [1 Mark]}$$

Step 4:

Since the slope of the tangent = slope of the chord, we have:

$$2(x - 2) = 2$$

$$\Rightarrow x - 2 = 1$$

$$\therefore x = 3 \text{ [1 Mark]}$$

Step 5:

When $x = 3$,

$$y = (3 - 2)^2 = 1 \text{ [1 Mark]}$$

Hence, the required point is $(3, 1)$.

9. Find the point on the curve $y = x^3 - 11x + 5$ at which the tangent is $y = x - 11$. [3 Marks]

Solution:**Step 1:**

The equation of the given curve is $y = x^3 - 11x + 5$.

Step 2:

The equation of the tangent to the given curve is given as $y = x - 11$ (which is of the form $y = mx + c$).

$$\therefore \text{Slope of the tangent} = 1 \text{ [}\frac{1}{2}\text{ Mark]}$$

Step 3:

Now,

the slope of the tangent to the given curve at the point (x, y) is given by,

$$y = x^3 - 11x + 5$$

On differentiating with respect to x , we get

$$\frac{dy}{dx} = 3x^2 - 11 \text{ [}\frac{1}{2}\text{ Mark]}$$

Step 4:

Then, we have:

$$3x^2 - 11 = 1$$

$$\Rightarrow 3x^2 = 12$$

$$\Rightarrow x^2 = 4$$

$$\Rightarrow x = \pm 2 \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 5:

When $x = 2$,

$$y = (2)^3 - 11(2) + 5$$

$$= 8 - 22 + 5 = -9. \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 6:

When $x = -2$,

$$y = (-2)^3 - 11(-2) + 5$$

$$= -8 + 22 + 5 = 19. \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 7:

Therefore, the required points are $(2, -9)$ and $(-2, 19)$. $\left[\frac{1}{2} \text{ Mark}\right]$

10. Find the equation of all lines having slope -1 that are tangents to the curve $y = \frac{1}{x-1}, x \neq 1$. **[4 Marks]**

Solution:**Step 1:**

The equation of the given curve is $y = \frac{1}{x-1}, x \neq 1$.

Step 2:

The slope of the tangents to the given curve at any point (x, y) is given by,

$$y = \frac{1}{x-1}$$

By differentiating with respect to x , we get

$$\frac{dy}{dx} = \frac{-1}{(x-1)^2} \quad \left[\mathbf{1Mark}\right]$$

Step 3:

Now,

If the slope of the tangent is -1 , then we have:

$$\frac{-1}{(x-1)^2} = -1$$

$$\Rightarrow (x-1)^2 = 1$$

$$\Rightarrow x-1 = \pm 1$$

$$\therefore x = 2 \text{ or } 0 \left[\frac{1}{2} \text{ Mark} \right]$$

Step 4:

When $x = 0, y = -1$ and

when $x = 2, y = 1$.

Thus, there are two tangents to the given curve having slope -1 .

These are passing through the points $(0, -1)$ and $(2, 1)$. $\left[1 \frac{1}{2} \text{ Mark} \right]$

Step 5:

So, the equation of the tangent through $(0, -1)$ is given by,

$$y - (-1) = 1 - (x - 0)$$

$$\Rightarrow y + 1 = -x$$

$$\Rightarrow y + x + 1 = 0 \left[\frac{1}{2} \text{ Mark} \right]$$

Step 6:

So, the equation of the tangent through $(2, 1)$ is given by,

$$y - 1 = -1(x - 2)$$

$$\Rightarrow y - 1 = -x + 2$$

$$\Rightarrow y + x - 3 = 0 \left[\frac{1}{2} \text{ Mark} \right]$$

Therefore, the equations of the required lines are $y + x + 1 = 0$ and $y + x - 3 = 0$.

11. Find the equation of all lines having slope 2 which are tangents to the curve $y = \frac{1}{x-3}, x \neq 3$.

[2 Marks]

Solution:

Step 1:

The equation of the given curve is $y = \frac{1}{x-3}, x \neq 3$.

Step 2:

The slope of the tangent to the given curve at any point (x, y) is given by,

$$y = \frac{1}{x-3}$$

By differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{-1}{(x-3)^2} \quad [1 \text{ Mark}]$$

Step 3:

Now,

If the slope of the tangent is 2, then we have:

$$\frac{-1}{(x-3)^2} = 2$$

$$\Rightarrow 2(x-3)^2 = -1$$

$$\Rightarrow (x-3)^2 = \frac{-1}{2} \quad [\frac{1}{2} \text{ Mark}]$$

Step 4:

This is not possible since the L.H.S. is positive while the R.H.S is negative.

Therefore, there is no tangent to the given curve which has slope 2. $[\frac{1}{2} \text{ Mark}]$

12. Find the equations of all lines having slope 0 which are tangent to the curve $y = \frac{1}{x^2-2x+3}$. [4 Marks]

Solution:

Step 1:

The equation of the given curve is $y = \frac{1}{x^2-2x+3}$.

Step 2:

The slope of the tangent to the given curve at any point (x, y) is given by,

$$y = \frac{1}{x^2-2x+3}$$

On differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{-(2x-2)}{(x^2-2x+3)^2} = \frac{-2(x-1)}{(x^2-2x+3)^2} \quad [1 \text{ mark}]$$

Step 3:

Now,

If the slope of the tangent is 0, then we have:

$$\frac{-2(x-1)}{(x^2-2x+3)^2} = 0$$

$$\Rightarrow -2(x-1) = 0$$

$\therefore x = 1$ [1 Mark]

Step 4:

When $x = 1$,

$y = \frac{1}{1-2+3} = \frac{1}{2}$ [1 Mark]

Step 5:

Hence, the equation of the tangent through $(1, \frac{1}{2})$ is given by,

$y - \frac{1}{2} = 0(x - 1)$

$\Rightarrow y - \frac{1}{2} = 0$

$\Rightarrow y = \frac{1}{2}$ [1 Mark]

Therefore, the equation of the required line is $y = \frac{1}{2}$

13. Find points on the curve $\frac{x^2}{9} + \frac{y^2}{16} = 1$ at which the tangents are [4 Marks]

(i) parallel to x -axis

(ii) parallel to y -axis

Solution:

Step 1:

The equation of the given curve is $\frac{x^2}{9} + \frac{y^2}{16} = 1$.

Step 2:

On differentiating both sides with respect to x , we have:

$\frac{2x}{9} + \frac{2y}{16} \cdot \frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-16x}{9y}$ [1 Mark]

Step 3:

(i). The tangent is parallel to the x -axis if the slope of the tangent $\frac{-16x}{9y} = 0$,

This is only possible if $x = 0$. [$\frac{1}{2}$ Mark]

Step 4:

So, for $x = 0$

Then, $\frac{x^2}{9} + \frac{y^2}{16} = 1$

$$\frac{0}{9} + \frac{y^2}{16} = 1$$

$$\Rightarrow y^2 = 16$$

$$\Rightarrow y = \pm 4$$

Therefore, the points at which the tangents are parallel to the x-axis are (0,4) and (0, -4). [1 Mark]

Step 5:

(ii). The tangent is parallel to the y-axis if the slope of the normal is 0, which

gives $\frac{-1}{\left(\frac{-16x}{9y}\right)} = \frac{9y}{16x} = 0$

$$\Rightarrow y = 0$$

So, for $y = 0$ [$\frac{1}{2}$ Mark]

Step 6:

Then, $\frac{x^2}{9} + \frac{y^2}{16} = 1$

$$\frac{x^2}{9} + \frac{0}{16} = 1$$

$$\Rightarrow x = \pm 3$$

Therefore, the points at which the tangents are parallel to the y-axis are (3,0) and (-3,0). [1 Mark]

14. Find the equations of the tangent and normal to the given curves at the indicated points:

(i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (0, 5) [4 Marks]

(ii) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (1, 3) [4 Marks]

(iii) $y = x^3$ at (1, 1) [4 Marks]

(iv) $y = x^2$ at (0, 0) [4 Marks]

(v) $x = \cos t, y = \sin t$ at $t = \frac{\pi}{4}$ [4 Marks]

Solution:

(i) **Step 1:**

Given:

The equation of the given curve is $y = x^4 - 6x^3 + 13x^2 - 10x + 5$.

Step 2:

On differentiating with respect to x , we get:

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$

$$\left. \frac{dy}{dx} \right|_{(0,5)} = -10$$

Hence, the slope of the tangent at $(0,5)$ is -10 . [1 Mark]

Step 3:

The equation of the tangent is given as:

$$y - 5 = -10(x - 0)$$

$$\Rightarrow y - 5 = -10x$$

$$\Rightarrow 10x + y = 5 \text{ [1 Mark]}$$

Step 4:

∴ The slope of the normal at $(0,5)$ is $\frac{-1}{\text{Slope of the tangent at } (0,5)} = \frac{1}{10}$ [1 Mark]

Step 5:

Thus, the equation of the normal at $(0,5)$ is given as:

$$y - 5 = \frac{1}{10}(x - 0)$$

$$\Rightarrow 10y - 50 = x$$

$$\Rightarrow x - 10y + 50 = 0 \text{ [1 Mark]}$$

Therefore, the required equations of the tangent and normal are $10x + y = 5$ and $x - 10y + 50 = 0$ respectively.

(i) **Step 1:**

Given:

The equation of the given curve is $y = x^4 - 6x^3 + 13x^2 - 10x + 5$.

Step 2:

On differentiating with respect to x , we get:

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$

$$\left. \frac{dy}{dx} \right|_{(1,3)} = 4 - 18 + 26 - 10 = 2$$

Hence, the slope of the tangent at $(1,3)$ is 2 . [1 Mark]

Step 3:

The equation of the tangent is given as:

$$y - 3 = 2(x - 1)$$

$$\Rightarrow y - 3 = 2x - 2$$

$$\Rightarrow y = 2x + 1 \quad \text{[1 Mark]}$$

Step 4:

$$\therefore \text{The slope of the normal at } (1,3) \text{ is } \frac{-1}{\text{Slope of the tangent at } (1,3)} = \frac{-1}{2} \quad \text{[1 Mark]}$$

Step 5:

Thus, the equation of the normal at (1,3) is given as:

$$y - 3 = -\frac{1}{2}(x - 1)$$

$$\Rightarrow 2y - 6 = x - 1$$

$$\Rightarrow x + 2y - 7 = 0 \quad \text{[1 Mark]}$$

Therefore, the required equations of the tangent and normal are $y = 2x + 1$ and $x + 2y - 7 = 0$ respectively.

(ii) **Step 1:**

Given: The equation of the given curve is $y = x^3$.

On differentiating with respect to x , we get:

$$\frac{dy}{dx} = 3x^2$$

$$\left. \frac{dy}{dx} \right|_{(1,1)} = 3(1)^2 = 3$$

Thus, the slope of the tangent at (1,1) is 3. **[1 Mark]**

Step 2:

The equation of the tangent is given as:

$$y - 1 = 3(x - 1)$$

$$\Rightarrow y = 3x - 2 \quad \text{[1 Mark]}$$

Step 3:

$$\therefore \text{The slope of the normal at } (1,1) \text{ is } \frac{-1}{\text{Slope of the tangent at } (1,1)} = \frac{-1}{3}. \quad \text{[1 Mark]}$$

Step 4:

Hence, the equation of the normal at (1,1) is given as:

$$y - 1 = -\frac{1}{3}(x - 1)$$

$$\Rightarrow 3y - 3 = -x + 1$$

$$\Rightarrow x + 3y - 4 = 0 \quad \text{[1 Mark]}$$

Therefore, the required equations of the tangent and normal are $y = 3x - 2$ and $x + 3y - 4 = 0$ respectively.

(iv) **Step 1:**

Given: The equation of the given curve is $y = x^2$.

On differentiating with respect to x , we get:

$$\frac{dy}{dx} = 2x$$

$$\left. \frac{dy}{dx} \right|_{(0,0)} = 0$$

Thus, the slope of the tangent at $(0,0)$ is 0. **[1 Mark]**

Step 2:

The equation of the tangent is given as:

$$y - 0 = 0(x - 0)$$

$$\Rightarrow y = 0 \text{ [1 Mark]}$$

Step 3:

$$\therefore \text{The slope of the normal at } (0,0) \text{ is } \frac{-1}{\text{Slope of the tangent at } (0,0)} = -\frac{1}{0} \text{ [1 Mark]}$$

Step 4:

which is not defined.

Hence, the equation of the normal at $(x_0, y_0) = (0,0)$ is given by $x = x_0 = 0$.

Therefore, the required equations of the tangent and normal are $y = 0$ and $x = 0$ respectively. **[1 Mark]**

(v) **Step 1:**

Given: The equation of the given curve is $x = \cos t, y = \sin t$.

$$x = \cos t \text{ and } y = \sin t$$

On differentiating with respect to t , we get:

$$\frac{dx}{dt} = -\sin t$$

$$\frac{dy}{dt} = \cos t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\cos t}{-\sin t} = -\cot t \text{ [1 Mark]}$$

Step 2:

$$\left. \frac{dy}{dx} \right|_{t=\frac{\pi}{4}} = -\cot t$$

$$= -\cot \frac{\pi}{4} = -1$$

The slope of the tangent at $t = \frac{\pi}{4}$ [1 Mark]

Step 3:

When $t = \frac{\pi}{4}$,

$$x = \frac{1}{\sqrt{2}} \text{ and } y = \frac{1}{\sqrt{2}}.$$

Hence, the slope of the tangent at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is -1 .

Thus, the equation of the tangent to the given curve at $t = \frac{\pi}{4}$ i.e., at $\left[\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right]$ is

$$y - \frac{1}{\sqrt{2}} = -1 \left(x - \frac{1}{\sqrt{2}}\right).$$

$$\Rightarrow x + y - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$$

$$\Rightarrow x + y - \sqrt{2} = 0 \text{ [1 Mark]}$$

Step 4:

\therefore The slope of the normal at $t = \frac{\pi}{4}$ is $\frac{-1}{\text{Slope of the tangent at } t=\frac{\pi}{4}} = 1$.

Hence, the equation of the normal to the given curve at $t = \frac{\pi}{4}$ i.e., at $\left[\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right]$ is

$$y - \frac{1}{\sqrt{2}} = 1 \left(x - \frac{1}{\sqrt{2}}\right).$$

$$\Rightarrow x = y \text{ [1 Mark]}$$

Therefore, the required equations of the tangent and normal are $x + y - \sqrt{2} = 0$ and $x = y$ respectively.

15. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$ which is

(a) parallel to the line $2x - y + 9 = 0$ [3 Marks]

(b) perpendicular to the line $5y - 15x = 13$ [4 Marks]

Solution:

Step 1:

Given:

The equation of the given curve is $y = x^2 - 2x + 7$.

Step 2:

On differentiating with respect to x , we get:

$$\frac{dy}{dx} = 2x - 2$$

Step 3:

(a) The equation of the line is $2x - y + 9 = 0$.

$$2x - y + 9 = 0$$

$$\therefore y = 2x + 9$$

This is of the form $y = mx + c$.

\therefore Slope of the line = 2 [1 Mark]

Step 4:

If a tangent is parallel to the line $2x - y + 9 = 0$, then the slope of the tangent is equal to the slope of the line.

Hence, we have:

$$2 = 2x - 2$$

$$\Rightarrow 2x = 4$$

$$\Rightarrow x = 2 \text{ [1 Mark]}$$

Step 5:

Now, $x = 2$ we get:

$$\Rightarrow y = -4 + 7 = 7$$

Thus, the equation of the tangent passing through (2,7) is given by,

$$y - 7 = 2(x - 2)$$

$$\Rightarrow y - 2x - 3 = 0 \text{ [1 Mark]}$$

Therefore, the equation of the tangent line to the given curve which is parallel to line $2x - y + 9 = 0$ is $y - 2x - 3 = 0$.

(b) Step 1:

Given:

The equation of the line is $5y - 15x = 13$.

$$5y - 15x = 13$$

$$\therefore y = 3x + \frac{13}{5}$$

This is of the form $y = mx + c$.

\therefore Slope of the line = 3 [1 Mark]

Step 2:

If a tangent is perpendicular to the line $5y - 15x = 13$, then the slope of the tangent is

$$\frac{-1}{\text{slope of the line}} = \frac{-1}{3}$$

$$\Rightarrow 2x - 2 = \frac{-1}{3}$$

$$\Rightarrow 2x = \frac{-1}{3} + 2$$

$$\Rightarrow 2x = \frac{5}{3}$$

$$\Rightarrow x = \frac{5}{6} \text{ [1 Mark]}$$

Step 3:

Now, $x = \frac{5}{6}$ we get:

$$\Rightarrow y = \frac{25}{36} - \frac{10}{6} + 7$$

$$y = \frac{25-60+252}{36} = \frac{217}{36} \text{ [1 Mark]}$$

Step 4:

Thus, the equation of the tangent passing through $(\frac{5}{6}, \frac{217}{36})$ is given by,

$$y - \frac{217}{36} = -\frac{1}{3}\left(x - \frac{5}{6}\right)$$

$$\Rightarrow \frac{36y - 217}{36} = \frac{-1}{18}(6x - 5)$$

$$\Rightarrow 36y - 217 = -2(6x - 5)$$

$$\Rightarrow 36y - 217 = -12x + 10$$

$$\Rightarrow 36y + 12x - 227 = 0 \text{ [1 Mark]}$$

Therefore, the equation of the tangent line to the given curve which is perpendicular to line $5y - 15x = 13$ is $36y + 12x - 227 = 0$.

16. Show that the tangents to the curve $y = 7x^3 + 11$ at the points where $x = 2$ and $x = -2$ are parallel. [2 Marks]

Solution:

Step 1:

The equation of the given curve is $y = 7x^3 + 11$.

Step 2:

On differentiating with respect to x , we get:

$$\therefore \frac{dy}{dx} = 21x^2 \text{ [1 Mark]}$$

Step 3:

The slope of the tangent to a curve at (x_0, y_0) is $\left. \frac{dy}{dx} \right|_{(x_0, y_0)}$

Therefore, the slope of the tangent at the point where $x = 2$ is given by,

$$\left. \frac{dy}{dx} \right|_{x=2} = 21x^2 = 21(2)^2 = 84$$

the slope of the tangent at the point where $x = -2$ is given by,

$$\left. \frac{dy}{dx} \right|_{x=-2} = 21x^2 = 21(-2)^2 = 84$$

It is observed that the slopes of the tangents at the points where $x = 2$ and $x = -2$ are equal.

Thus, tangent at $x = 2$ and tangent at $x = -2$ are parallel. **[1 Mark]**

Hence, it is proved that the two tangents are parallel.

17. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y -coordinate of the point. **[4 Marks]**

Solution:

Step 1:

The equation of the given curve is $y = x^3$.

Step 2:

On differentiating with respect to x , we get:

$$\therefore \frac{dy}{dx} = 3x^2$$

Now,

The slope of the tangent at the point (x, y) is given by,

$$\left. \frac{dy}{dx} \right|_{(x, y)} = 3x^2 \text{ [1 Mark]}$$

Step 3:

When the slope of the tangent is equal to the y -coordinate of the point, then $y = 3x^2$. **[1 Mark]**

Step 4:

Also, we have $y = x^3$.

$$\therefore 3x^2 = x^3$$

$$x^2(x - 3) = 0$$

$$x = 0, x = 3 \text{ [1 Mark]}$$

Step 5:

When $x = 0$,

$$\Rightarrow y = 0$$

When $x = 3$,

$$\Rightarrow y = 3(3)^2 = 27$$

Hence, the required points are $(0,0)$ and $(3,27)$. [1 Mark]

18. For the curve $y = 4x^3 - 2x^5$, find all the points at which the tangents passes through the origin.

[4 Marks]

Solution:

Step 1:

Given:

The equation of the given curve is $y = 4x^3 - 2x^5$

Step 2:

On differentiating with respect to x , we get:

$$\therefore \frac{dy}{dx} = 12x^2 - 10x^4$$

Thus, the slope of the tangent at a point (x, y) is $12x^2 - 10x^4$. [1 Mark]

Step 3:

The equation of the tangent at (x, y) is given by,

$$Y - y = (12x^2 - 10x^4)(X - x) \quad \dots (1)$$

When the tangent passes through the origin $(0,0)$, then $X = Y = 0$.

Therefore, equation (1) reduces to:

$$-y = (12x^2 - 10x^4)(-x)$$

$$y = 12x^3 - 10x^5 \text{ [1 Mark]}$$

Step 4:

Also, we have $y = 4x^3 - 2x^5$

$$\therefore 12x^3 - 10x^5 = 4x^3 - 2x^5$$

$$\Rightarrow 8x^5 - 8x^3 = 0$$

$$\Rightarrow x^5 - x^3 = 0$$

$$\Rightarrow x^3(x^2 - 1) = 0$$

$$\Rightarrow x = 0, x = \pm 1 \text{ [1 Mark]}$$

Step 5:

When $x = 0$,

$$y = 4(0)^3 - 2(0)^5 = 0$$

When $x = 1$,

$$y = 4(1)^3 - 2(1)^5 = 2$$

When $x = -1$,

$$y = 4(-1)^3 - 2(-1)^5 = -2$$

Hence, the required points are $(0,0)$, $(1,2)$, and $(-1,-2)$. [1 Mark]

19. Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to the x -axis. [2 Marks]

Solution:

Step 1:

The equation of the given curve is $x^2 + y^2 - 2x - 3 = 0$.

Step 2:

On differentiating with respect to x , we have:

$$2x + 2y \frac{dy}{dx} - 2 = 0$$

$$\Rightarrow y \frac{dy}{dx} = 1 - x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-x}{y} \text{ [1 Mark]}$$

Step 3:

Now, the tangents are parallel to the x -axis if the slope of the tangent is 0.

$$\therefore \frac{1-x}{y} = 0$$

$$\Rightarrow 1 - x = 0$$

$$\Rightarrow x = 1 \left[\frac{1}{2} \text{ Mark} \right]$$

But, $x^2 + y^2 - 2x - 3 = 0$ for $x = 1$.

$$y^2 = 4$$

$$\Rightarrow y = \pm 2$$

Therefore, the points at which the tangents are parallel to the x -axis are $(1, 2)$ and $(1, -2)$.
 $\left[\frac{1}{2} \text{ Mark} \right]$

20. Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$. [4 Marks]

Solution:

Step 1:

Given:

The equation of the given curve is $ay^2 = x^3$.

Step 2:

On differentiating with respect to x , we have:

$$2ay \frac{dy}{dx} = 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{2ay} \quad [1 \text{ Mark}]$$

Step 3:

The slope of a tangent to the curve at (x_0, y_0) is $\left. \frac{dy}{dx} \right|_{(x_0, y_0)}$

Hence, the slope of the tangent to the given curve is,

$$\left. \frac{dy}{dx} \right|_{(am^2, am^3)} = \frac{3(am^2)^2}{2a(am^3)}$$

$$= \frac{3a^2m^4}{2a^2m^3} = \frac{3m}{2} \quad [1 \text{ Mark}]$$

Step 4:

$$\therefore \text{Slope of normal at } (am^2, am^3) = \frac{-1}{\text{slope of the tangent at } (am^2, am^3)} = \frac{-2}{3m} \quad [1 \text{ Mark}]$$

Step 5:

Hence, the equation of the normal at (am^2, am^3) is given by,

$$y - am^3 = \frac{-2}{3m}(x - am^2)$$

$$\Rightarrow 3my - 3am^4 = -2x + 2am^2$$

$$\Rightarrow 2x + 3my - am^2(2 + 3m^2) = 0 \text{ [1 Mark]}$$

Therefore, the equation of the normal is $2x + 3my - am^2(2 + 3m^2) = 0$

21. Find the equation of the normal to the curve $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$. [6 Marks]

Solution:

Step 1:

The equation of the given curve is $y = x^3 + 2x + 6$.

Step 2:

The slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx} = 3x^2 + 2 \text{ [1 Mark]}$$

Step 3:

\therefore Slope of the normal to the given curve at any point $(x, y) = \frac{-1}{\text{slope of the tangent at } (x,y)}$

$$= \frac{-1}{3x^2+2} \text{ [1 Mark]}$$

Step 4:

The equation of the given line is $x + 14y + 4 = 0$.

$$x + 14y + 4 = 0 \quad (\because \text{which is of the form } y = mx + c)$$

$$y = -\frac{1}{14}x - \frac{4}{14}$$

\therefore Slope of the given line = $\frac{-1}{14}$ [1 Mark]

Step 5:

If the normal is parallel to the line, then we must have the slope of the normal being equal to the slope of the line.

$$\therefore \frac{-1}{3x^2 + 2} = \frac{-1}{14}$$

$$\Rightarrow 3x^2 + 2 = 14$$

$$\Rightarrow 3x^2 = 12$$

$$\Rightarrow x^2 = 4$$

$$\Rightarrow x = \pm 2 \text{ [1 Mark]}$$

Step 6:

When $x = 2$,

$$y = 8 + 4 + 6 = 18.$$

When $x = -2$,

$$y = -8 - 4 + 6 = -6.$$

Therefore, there are two normal to the given curve with slope $\frac{-1}{14}$ and passing through the points $(2,18)$ and $(-2, -6)$. **[1 Mark]**

Step 7:

Thus, the equation of the normal through $(2,18)$ is given by,

$$y - 18 = \frac{-1}{14}(x - 2)$$

$$\Rightarrow 14y - 252 = -x + 2$$

$$\Rightarrow x + 14y - 254 = 0$$

And, the equation of the normal through $(-2, -6)$ is given by,

$$y - (-6) = \frac{-1}{14}[x - (-2)]$$

$$\Rightarrow y + 6 = \frac{-1}{14}(x + 2)$$

$$\Rightarrow 14y + 84 = -x - 2$$

$$\Rightarrow x + 14y + 86 = 0 \quad \mathbf{[1 Mark]}$$

Therefore, the equations of the normal to the given curve which are parallel to the given line are $x + 14y - 254 = 0$ and $x + 14y + 86 = 0$.

22. Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

[4 Marks]

Solution:

Step 1:

The equation of the given parabola is $y^2 = 4ax$.

Step 2:

On differentiating $y^2 = 4ax$ with respect to x , we have:

$$2y \frac{dy}{dx} = 4a$$

$$\Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

∴ The slope of the tangent at $(at^2, 2at)$ is $\left. \frac{dy}{dx} \right|_{(at^2, 2at)} = \frac{2a}{2at} = \frac{1}{t}$ [1 Mark]

Step 3:

Thus, the equation of the tangent at $(at^2, 2at)$ is given by,

$$y - 2at = \frac{1}{t}(x - at^2)$$

$$\Rightarrow ty - 2at^2 = x - at^2$$

$$\Rightarrow ty = x + at^2 \quad \text{[1 Mark]}$$

Step 3:

Now, the slope of the normal at $(at^2, 2at) = \frac{-1}{\text{Slope of the tangent at } (at^2, 2at)} = -t$ [1 Mark]

Step 4:

Thus, the equation of the normal at $(at^2, 2at)$ is given as:

$$y - 2at = -t(x - at^2)$$

$$\Rightarrow y - 2at = -tx + at^3$$

$$\Rightarrow y = -tx + 2at + at^3 \quad \text{[1 Mark]}$$

Therefore, the required equation of the tangent and normal are $ty = x + at^2$ and $y = -tx + 2at + at^3$ respectively.

23. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$.

[Hint: Two curves intersect at right angle if the tangents to the curves at the point of intersection are perpendicular to each other.] [4 Marks]

Solution:

Step 1:

The equations of the given curves are given as $x = y^2$ and $xy = k$.

Step 2:

Putting $x = y^2$ in $xy = k$, we get:

$$y^3 = k$$

$$\Rightarrow y = k^{\frac{1}{3}}$$

Now, substituting y in $x = y^2$, we get:

$$x = k^{\frac{2}{3}}$$

Thus, the point of intersection of the given curves is $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$. [1 Mark]

Step 3:

Differentiating $x = y^2$ with respect to x , we have:

$$1 = 2y \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2y}$$

Hence, the slope of the tangent to the curve $x = y^2$ at $(k^{\frac{2}{3}}, k^{\frac{1}{3}})$

$$\left. \frac{dy}{dx} \right|_{(k^{\frac{2}{3}}, k^{\frac{1}{3}})} = \frac{1}{2k^{\frac{1}{3}}} \quad \text{[1 Mark]}$$

Step 4:

On differentiating $xy = k$ with respect to x , we have:

$$x \frac{dy}{dx} + y = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

\therefore Slope of the tangent to the curve $xy = k$ at $(k^{\frac{2}{3}}, k^{\frac{1}{3}})$ is given by,

$$\left. \frac{dy}{dx} \right|_{(k^{\frac{2}{3}}, k^{\frac{1}{3}})} = \left. \frac{-y}{x} \right|_{(k^{\frac{2}{3}}, k^{\frac{1}{3}})}$$

$$= -\frac{k^{\frac{1}{3}}}{k^{\frac{2}{3}}} = \frac{-1}{k^{\frac{1}{3}}} \quad \text{[1 Mark]}$$

Step 5:

We know that two curves intersect at right angles, if the tangents to the curves at the point of intersection i.e., at $(k^{\frac{2}{3}}, k^{\frac{1}{3}})$ are perpendicular to each other.

This implies that we should have the product of the tangents as -1 .

Hence, the given two curves cut at right angles if the product of the slopes of their respective tangents at $(k^{\frac{2}{3}}, k^{\frac{1}{3}})$ is -1 .

$$\Rightarrow (\text{Slope of tangent to the curve } x = y^2) \times (\text{Slope of tangent to the curve } xy = k) = -1$$

$$\text{i.e., } \left(\frac{1}{2k^{\frac{1}{3}}} \right) \left(\frac{-1}{k^{\frac{1}{3}}} \right) = -1$$

$$\frac{1}{2k^{\frac{1}{3}} \times k^{\frac{1}{3}}} = -1$$

$$\frac{1}{2k^{\frac{1}{3} + \frac{1}{3}}} = 1$$

$$\frac{1}{2k^{\frac{2}{3}}} = 1$$

$$\Rightarrow 2k^{\frac{2}{3}} = 1$$

$$\Rightarrow \left(2k^{\frac{2}{3}}\right)^3 = (1)^2$$

$$\Rightarrow 8k^2 = 1 \quad \text{[1 Mark]}$$

Hence, the given two curves cut at right angles if $8k^2 = 1$.

24. Find the equations of the tangent and normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0) .

[3 Marks]

Solution:

Step 1:

The given equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Step 2:

We know that, slope of tangent is $\frac{dy}{dx}$

On differentiating $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ we get:

$$\frac{2x}{a^2} - \frac{2y}{b^2} \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{2y}{b^2} \frac{dy}{dx} = \frac{2x}{a^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{b^2x}{a^2y}$$

Hence, the slope of the tangent at (x_0, y_0) is $\left. \frac{dy}{dx} \right|_{(x_0, y_0)} = \frac{b^2x_0}{a^2y_0}$ [1 Mark]

Step 3:

Then, the equation of the tangent at (x_0, y_0)

$$y - y_0 = \frac{b^2x_0}{a^2y_0}(x - x_0)$$

$$a^2y_0(y - y_0) = b^2x_0(x - x_0)$$

$$\Rightarrow a^2yy_0 - a^2y_0^2 = b^2xx_0 - b^2x_0^2$$

$$\Rightarrow b^2xx_0 - a^2yy_0 - b^2x_0^2 + a^2y_0^2 = 0$$

$$\Rightarrow \frac{xx_0}{a^2} - \frac{yy_0}{b^2} - \left(\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2}\right) = 0 \quad \text{[On dividing both sides by } a^2b^2]$$

$$\Rightarrow \frac{xx_0}{a^2} - \frac{yy_0}{b^2} - 1 = 0 \quad \left[(x_0, y_0) \text{ lies on the hyperbola } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \right]$$

$$\Rightarrow \frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1 \quad [1\frac{1}{2} \text{ Marks}]$$

Step 4:

Now, the slope of the normal at (x_0, y_0) is given by,

$$\frac{-1}{\text{Slope of the tangent at } (x_0, y_0)} = \frac{-a^2 y_0}{b^2 x_0} [1 \text{ Marks}]$$

Step 5:

Hence, the equation of the normal at (x_0, y_0) is given by,

$$y - y_0 = \frac{-a^2 y_0}{b^2 x_0} (x - x_0)$$

$$\Rightarrow \frac{y - y_0}{a^2 y_0} = \frac{-(x - x_0)}{b^2 x_0}$$

$$\Rightarrow \frac{y - y_0}{a^2 y_0} + \frac{(x - x_0)}{b^2 x_0} = 0 \quad [1 \text{ Mark}]$$

Therefore, the required equations of the tangent and normal are $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$ and $\frac{y - y_0}{a^2 y_0} + \frac{(x - x_0)}{b^2 x_0} = 0$ respectively.

25. Find the equation of the tangent to the curve $y = \sqrt{3x - 2}$ which is parallel to the line $4x - 2y + 5 = 0$. [4 Marks]

Solution:

Step 1:

The equation of the given curve is $y = \sqrt{3x - 2}$

Step 2:

The slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx} = \frac{3}{2\sqrt{3x-2}} \quad [1 \text{ Mark}]$$

Step 3:

The equation of the given line is $4x - 2y + 5 = 0$.

$$4x - 2y + 5 = 0$$

$$\therefore y = 2x + \frac{5}{2} \quad (\text{which is of the form } y = mx + c)$$

$$\therefore \text{Slope of the line} = 2$$

Now, the tangent to the given curve is parallel to the line $4x - 2y - 5 = 0$ if the slope of the tangent is equal to the slope of the line. [1 Mark]

Step 4:

$$\frac{3}{2\sqrt{3x-2}} = 2$$

$$\Rightarrow \sqrt{3x-2} = \frac{3}{4}$$

$$\Rightarrow 3x-2 = \frac{9}{16}$$

$$\Rightarrow 3x = \frac{9}{16} + 2 = \frac{41}{16}$$

$$\Rightarrow x = \frac{41}{48} \text{ [1 Mark]}$$

Step 5:

When $x = \frac{41}{48}$,

$$y = \sqrt{3\left(\frac{41}{48}\right) - 2} = \sqrt{\frac{41}{16} - 2} = \sqrt{\frac{41-32}{16}} = \sqrt{\frac{9}{16}} = \frac{3}{4}$$

Hence, the equation of tangent at point $\left(\frac{41}{48}, \frac{3}{4}\right)$ is given by

$$y - \frac{3}{4} = 2\left(x - \frac{41}{48}\right)$$

$$\Rightarrow \frac{4y - 3}{4} = 2\left(\frac{48x - 41}{48}\right)$$

$$\Rightarrow 4y - 3 = \frac{48x - 41}{6}$$

$$\Rightarrow 24y - 18 = 48x - 41$$

$$\Rightarrow 48x - 24y = 23 \text{ [1 Mark]}$$

Therefore, the equation of the required tangent is $48x - 24y = 23$.

26. The slope of the normal to the curve $y = 2x^2 + 3 \sin x$ at $x = 0$ is [2 Marks]

- (A) 3
- (B) $\frac{1}{3}$
- (C) -3
- (D) $-\frac{1}{3}$

Solution:

Step 1:

The equation of the given curve is $y = 2x^2 + 3 \sin x$.

Step 2:

The slope of the tangent to the given curve at $x = 0$ is given by,

$$\begin{aligned} \left. \frac{dy}{dx} \right|_{x=0} &= 4x + 3 \cos x \Big|_{x=0} \\ &= 0 + 3 \cos 0 = 3 \quad \text{[1 Mark]} \end{aligned}$$

Step 3:

Hence, the slope of the normal to the given curve at $x = 0$ is

$$\text{Slope of the tangent at } x = 0 = \frac{-1}{3}$$

Therefore, the required slope of normal to the given curve is $\frac{-1}{3}$

So, the correct answer is D. [1 Mark]

27. The line $y = x + 1$ is a tangent to the curve $y^2 = 4x$ at the point [4 Marks]

- (A) (1, 2)
- (B) (2, 1)
- (C) (1, -2)
- (D) (-1, 2)

Solution:

Step 1:

The equation of the given curve is $y^2 = 4x$.

Step 2:

On differentiating with respect to x , we have:

$$\begin{aligned} 2y \frac{dy}{dx} &= 4 \\ \Rightarrow \frac{dy}{dx} &= \frac{2}{y} \end{aligned}$$

Hence, the slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx} = \frac{2}{y} \quad \text{[1 Mark]}$$

Step 3:

The given line is $y = x + 1$ (which is of the form $y = mx + c$)

∴ Slope of the line = 1 [1 Mark]

Step 4:

The line $y = x + 1$ is a tangent to the given curve if the slope of the line is equal to the slope of the tangent. Also, the line must intersect the curve.

Thus, we must have:

$$\frac{2}{y} = 1$$

$$\Rightarrow y = 2 \text{ [1 Mark]}$$

Step 5:

$$\text{Now, } y = x + 1$$

$$\Rightarrow x = y - 1$$

$$= 2 - 1 = 1$$

Therefore, the line $y = x + 1$ is a tangent to the given curve at the point (1,2).

So, the correct answer is A. [1 Mark]

Exercise 6.4

1. Using differentials, find the approximate value of each of the following up to 3 places of decimal

(i) $\sqrt{25.3}$ [2 Marks]

(ii) $\sqrt{49.5}$ [2 Marks]

(iii) $\sqrt{0.6}$ [2 Marks]

(iv) $(0.009)^{\frac{1}{3}}$ [2 Marks]

(v) $(0.009)^{\frac{1}{10}}$ [2 Marks]

(vi) $(15)^{\frac{1}{4}}$ [2 Marks]

(vii) $(26)^{\frac{1}{3}}$ [2 Marks]

(viii) $(255)^{\frac{1}{4}}$ [2 Marks]

(ix) $(82)^{\frac{1}{4}}$ [2 Marks]

(x) $(401)^{\frac{1}{2}}$ [2 Marks]

(xi) $(0.0037)^{\frac{1}{2}}$ [2 Marks]

(xii) $(26.57)^{\frac{1}{3}}$ [2 Marks]

(xiii) $(81.5)^{\frac{1}{4}}$ [2 Marks]

(xiv) $(3.968)^{\frac{3}{2}}$ [2 Marks]

(xv) $(32.15)^{\frac{1}{5}}$ [2 Marks]

Solution:

(i) **Step 1:**

Given: $\sqrt{25.3}$

Step 2:

Consider $y = \sqrt{x}$.

Let $x = 25$ and $\Delta x = 0.3$. [$\frac{1}{2}$ Mark]

Step 3:

Then,

$$\begin{aligned}\Delta y &= \sqrt{x + \Delta x} - \sqrt{x} \\ &= \sqrt{25.3} - \sqrt{25} = \sqrt{25.3} - 5 \\ \Rightarrow \sqrt{25.3} &= \Delta y + 5 \quad \left[\frac{1}{2} \text{ Mark}\right]\end{aligned}$$

Step 4:

Since, dy is approximately equal to Δy and is given by,

$$\begin{aligned}dy &= \left(\frac{dy}{dx}\right) \Delta x \\ \Delta y &= \frac{1}{2\sqrt{x}}(0.3) \quad \left[\text{as } y = \sqrt{x}\right] \\ &= \frac{1}{2\sqrt{25}}(0.3) = 0.03 \quad \left[\frac{1}{2} \text{ Mark}\right]\end{aligned}$$

Step 5:

Now, by substituting the value of Δy we get:

$$\begin{aligned}\sqrt{25.3} &= \Delta y + 5 \\ \sqrt{25.3} &= 0.03 + 5 \\ \sqrt{25.3} &= 5.03 \quad \left[\frac{1}{2} \text{ Mark}\right]\end{aligned}$$

Therefore, the approximate value of $\sqrt{25.3}$ is 5.03.

(ii) **Step 1:**

Given: $\sqrt{49.5}$

Step 2:

Consider $y = \sqrt{x}$.

Let $x = 49$ and $\Delta x = 0.5$. $[\frac{1}{2} \text{ Mark}]$

Step 3:

Then,

$$\begin{aligned}\Delta y &= \sqrt{x + \Delta x} - \sqrt{x} \\ &= \sqrt{49.5} - \sqrt{49} = \sqrt{49.5} - 7 \\ \Rightarrow \sqrt{49.5} &= 7 + \Delta y \quad [\frac{1}{2} \text{ Mark}]\end{aligned}$$

Step 4:

Since, dy is approximately equal to Δy and is given by,

$$\begin{aligned}dy &= \left(\frac{dy}{dx}\right) \Delta x \\ \Delta y &= \frac{1}{2\sqrt{x}}(0.5) \quad [as \ y = \sqrt{x}] \\ &= \frac{1}{2\sqrt{49}}(0.5) = \frac{1}{14}(0.5) = 0.035 \quad [\frac{1}{2} \text{ Mark}]\end{aligned}$$

Step 5:

Now, by substituting the value of Δy we get:

$$\begin{aligned}\sqrt{49.5} &= 7 + \Delta y \\ \sqrt{49.5} &= 7 + 0.035 \\ \sqrt{49.5} &= 7.035 \quad [\frac{1}{2} \text{ Mark}]\end{aligned}$$

Therefore, the approximate value of $\sqrt{49.5}$ is 7.035.

(iii) **Step 1:**

Given: $\sqrt{0.6}$

Step 2:

Consider $y = \sqrt{x}$.

Let $x = 1$ and $\Delta x = -0.4$. $[\frac{1}{2} \text{ Mark}]$

Step 3:

Then,

$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x} = \sqrt{0.6} - 1$$

$$\Rightarrow \sqrt{0.6} = 1 + \Delta y \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 4:

Since, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right) \Delta x$$

$$\Delta y = \frac{1}{2\sqrt{x}} (\Delta x) \quad [\text{as } y = \sqrt{x}]$$

$$= \frac{1}{2\sqrt{1}} (-0.4) = -0.2 \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 5:

Now, by substituting the value of Δy we get:

$$\sqrt{0.6} = 1 + \Delta y$$

$$\sqrt{0.6} = 1 + (-0.2)$$

$$\sqrt{0.6} = 0.8 \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Hence, the approximate value of $\sqrt{0.6}$ is 0.8.

(iv) Step 1:

Given: $(0.009)^{\frac{1}{3}}$

Step 2:

Consider $y = x^{\frac{1}{3}}$.

Let $x = 0.008$ and $\Delta x = 0.001$. $\left[\frac{1}{2} \text{ Mark}\right]$

Step 3:

Then,

$$\Delta y = (x + \Delta x)^{\frac{1}{3}} - (x)^{\frac{1}{3}}$$

$$= (0.009)^{\frac{1}{3}} - (0.008)^{\frac{1}{3}}$$

$$= (0.009)^{\frac{1}{3}} - 0.2$$

$$\Rightarrow (0.009)^{\frac{1}{3}} = 0.2 + \Delta y \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 4:

Since, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right) \Delta x$$

$$\Delta y = \frac{1}{3(x)^{\frac{2}{3}}} (\Delta x) \quad \left[as\ y = x^{\frac{1}{3}}\right]$$

$$= \frac{1}{3 \times 0.04} (0.001) = \frac{0.001}{0.12} = 0.008 \quad \left[\frac{1}{2}\ \text{Mark}\right]$$

Step 5:

Now, by substituting the value of Δy we get:

$$(0.009)^{\frac{1}{3}} = 0.2 + \Delta y$$

$$(0.009)^{\frac{1}{3}} = 0.2 + 0.008$$

$$(0.009)^{\frac{1}{3}} = 0.208 \quad \left[\frac{1}{2}\ \text{Mark}\right]$$

Therefore, the approximate value of $(0.009)^{\frac{1}{3}}$ is 0.208.

(v) Step 1:

Given: $(0.009)^{\frac{1}{10}}$

Step 2:

Consider $y = (x)^{\frac{1}{10}}$.

Let $x = 1$ and $\Delta x = -0.001$. $\left[\frac{1}{2}\ \text{Mark}\right]$

Step 3:

Then,

$$\Delta y = (x + \Delta x)^{\frac{1}{10}} - (x)^{\frac{1}{10}} = (0.999)^{\frac{1}{10}} - 1$$

$$\Rightarrow (0.999)^{\frac{1}{10}} = 1 + \Delta y \quad \left[\frac{1}{2}\ \text{Mark}\right]$$

Step 4:

Since, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right) \Delta x$$

$$\Delta y = \frac{1}{10(x)^{\frac{9}{10}}} (\Delta x) \quad \left[as\ y = (x)^{\frac{1}{10}}\right]$$

$$= \frac{1}{10} (-0.001) = -0.0001 \quad \left[\frac{1}{2}\ \text{Mark}\right]$$

Step 5:

Now, by substituting the value of Δy we get:

$$(0.999)^{\frac{1}{10}} = 1 + \Delta y$$

$$(0.999)^{\frac{1}{10}} = 1 + (-0.0001)$$

$$(0.999)^{\frac{1}{10}} = 0.9999 \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Therefore, the approximate value of $(0.999)^{\frac{1}{10}}$ is 0.9999.

(vi) **Step 1:**

Given: $(15)^{\frac{1}{4}}$

Step 2:

Consider $y = x^{\frac{1}{4}}$.

Let $x = 16$ and $\Delta x = -1$. $\left[\frac{1}{2} \text{ Mark}\right]$

Step 3:

Then,

$$\begin{aligned} \Delta y &= (x + \Delta x)^{\frac{1}{4}} - x^{\frac{1}{4}} \\ &= (15)^{\frac{1}{4}} - (16)^{\frac{1}{4}} = (15)^{\frac{1}{4}} - 2 \\ \Rightarrow (15)^{\frac{1}{4}} &= 2 + \Delta y \quad \left[\frac{1}{2} \text{ Mark}\right] \end{aligned}$$

Step 4:

Since, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right) \Delta x$$

$$\Delta y = \frac{1}{4(x)^{\frac{3}{4}}} (\Delta x) \quad \left[\text{as } y = x^{\frac{1}{4}}\right]$$

$$= \frac{1}{4(16)^{\frac{3}{4}}} (-1) = \frac{-1}{4 \times 8} = \frac{-1}{32} = -0.03125 \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 5:

Now, by substituting the value of Δy we get:

$$(15)^{\frac{1}{4}} = 2 + \Delta y$$

$$(15)^{\frac{1}{4}} = 2 + (-0.03125)$$

$$(15)^{\frac{1}{4}} = 1.96875$$

Therefore, the approximate value of $(15)^{\frac{1}{4}}$ is 1.96875.

(vii) **Step 1:**

Given: $(26)^{\frac{1}{3}}$

Step 2:

Consider $y = (x)^{\frac{1}{3}}$.

Let $x = 27$ and $\Delta x = -1$. $[\frac{1}{2} \text{ Mark}]$

Step 3:

Then,

$$\begin{aligned}\Delta y &= (x + \Delta x)^{\frac{1}{3}} - (x)^{\frac{1}{3}} \\ &= (26)^{\frac{1}{3}} - (27)^{\frac{1}{3}} = (26)^{\frac{1}{3}} - 3 \\ \Rightarrow (26)^{\frac{1}{3}} &= 3 + \Delta y \quad [\frac{1}{2} \text{ Mark}]\end{aligned}$$

Step 4:

Since, dy is approximately equal to Δy and is given by,

$$\begin{aligned}dy &= \left(\frac{dy}{dx}\right) \Delta x \\ \Delta y &= \frac{1}{3(x)^{\frac{2}{3}}} (\Delta x) \quad \left[\text{as } y = (x)^{\frac{1}{3}}\right] \\ &= \frac{1}{3(24)^{\frac{2}{3}}} (-1) = \frac{-1}{27} = -0.0370 \quad [\frac{1}{2} \text{ Mark}]\end{aligned}$$

Step 5:

Now, by substituting the value of Δy we get:

$$(26)^{\frac{1}{3}} = 3 + \Delta y$$

$$(26)^{\frac{1}{3}} = 3 + (-0.0370)$$

$$(26)^{\frac{1}{3}} = 2.9629$$

Therefore, the approximate value of $(26)^{\frac{1}{3}}$ is 2.9629.

(viii) **Step 1:**

Given: $(255)^{\frac{1}{4}}$

Step 2:

Consider $y = (x)^{\frac{1}{4}}$.

Let $x = 256$ and $\Delta x = -1$. [$\frac{1}{2}$ Mark]

Step 3:

Then,

$$\begin{aligned}\Delta y &= (x + \Delta x)^{\frac{1}{4}} - (x)^{\frac{1}{4}} \\ &= (255)^{\frac{1}{4}} - (256)^{\frac{1}{4}} = (255)^{\frac{1}{4}} - 4 \\ \Rightarrow (255)^{\frac{1}{4}} &= 4 + \Delta y \quad \left[\frac{1}{2} \text{ Mark}\right]\end{aligned}$$

Step 4:

Since, dy is approximately equal to Δy and is given by,

$$\begin{aligned}dy &= \left(\frac{dy}{dx}\right) \Delta x \\ \Delta y &= \frac{1}{4(x)^{\frac{3}{4}}} (\Delta x) \quad \left[\text{as } y = x^{\frac{1}{4}}\right] \\ &= \frac{1}{4(256)^{\frac{3}{4}}} (-1) = \frac{-1}{4 \times 4^3} = -0.0039 \quad \left[\frac{1}{2} \text{ Mark}\right]\end{aligned}$$

Step 5:

Now, by substituting the value of Δy we get:

$$\begin{aligned}\therefore (255)^{\frac{1}{4}} &= 4 + \Delta y \\ (255)^{\frac{1}{4}} &= 4 + (-0.0039) \\ (255)^{\frac{1}{4}} &= 3.9961 \quad \left[\frac{1}{2} \text{ Mark}\right]\end{aligned}$$

Hence, the approximate value of $(255)^{\frac{1}{4}}$ is 3.9961.

(ix) **Step 1:**

Given: $(82)^{\frac{1}{4}}$

Step 2:

Consider $y = x^{\frac{1}{4}}$.

Let $x = 81$ and $\Delta x = 1$. $[\frac{1}{2} \text{ Mark}]$

Step 3:

Then,

$$\begin{aligned} \Delta y &= (x + \Delta x)^{\frac{1}{4}} - (x)^{\frac{1}{4}} \\ &= (82)^{\frac{1}{4}} - (81)^{\frac{1}{4}} = (82)^{\frac{1}{4}} - 3 \\ \Rightarrow (82)^{\frac{1}{4}} &= 3 + \Delta y \quad [\frac{1}{2} \text{ Mark}] \end{aligned}$$

Step 4:

Since, dy is approximately equal to Δy and is given by,

$$\begin{aligned} dy &= \left(\frac{dy}{dx}\right) \Delta x \\ \Delta y &= \frac{1}{4(x)^{\frac{3}{4}}} (\Delta x) \quad \left[\text{as } y = x^{\frac{1}{4}} \right] \\ &= \frac{1}{4(81)^{\frac{3}{4}}} (1) = \frac{1}{4(3)^3} = \frac{1}{108} = 0.009 \quad [\frac{1}{2} \text{ Mark}] \end{aligned}$$

Step 5:

Now, by substituting the value of Δy we get:

$$\begin{aligned} (82)^{\frac{1}{4}} &= 3 + \Delta y \\ (82)^{\frac{1}{4}} &= 3 + 0.009 \\ (82)^{\frac{1}{4}} &= 3.009 \quad [\frac{1}{2} \text{ Mark}] \end{aligned}$$

Therefore, the approximate value of $(82)^{\frac{1}{4}}$ is 3.009.

(x)**Step 1:**

Given: $(401)^{\frac{1}{2}}$

Step 2:

Consider $y = x^{\frac{1}{2}}$.

Let $x = 400$ and $\Delta x = 1$. $[\frac{1}{2} \text{ Mark}]$

Step 3:

Then,

$$\begin{aligned}\Delta y &= \sqrt{x + \Delta x} - \sqrt{x} \\ &= \sqrt{401} - \sqrt{400} = \sqrt{401} - 20 \\ \Rightarrow \sqrt{401} &= 20 + \Delta y \quad \left[\frac{1}{2} \text{ Mark}\right]\end{aligned}$$

Step 4:

Since, dy is approximately equal to Δy and is given by,

$$\begin{aligned}dy &= \left(\frac{dy}{dx}\right) \Delta x \\ \Delta y &= \frac{1}{2\sqrt{x}} (\Delta x) \quad \left[\text{as } y = x^{\frac{1}{2}}\right] \\ &= \frac{1}{2 \times 20} (1) = \frac{1}{40} = 0.025 \quad \left[\frac{1}{2} \text{ Mark}\right]\end{aligned}$$

Step 5:

Now, by substituting the value of Δy we get:

$$\begin{aligned}\therefore \sqrt{401} &= 20 + \Delta y \\ \sqrt{401} &= 20 + 0.025 \\ \sqrt{401} &= 20.025 \quad \left[\frac{1}{2} \text{ Mark}\right]\end{aligned}$$

Hence, the approximate value of $\sqrt{401}$ is 20.025.

(xi) **Step 1:**

Given: $(0.0037)^{\frac{1}{2}}$

Step 2:

Consider $y = x^{\frac{1}{2}}$.

Let $x = 0.0036$ and $\Delta x = 0.0001$. $\left[\frac{1}{2} \text{ Mark}\right]$

Step 3:

Then,

$$\begin{aligned}\Delta y &= (x + \Delta x)^{\frac{1}{2}} - (x)^{\frac{1}{2}} \\ &= (0.0037)^{\frac{1}{2}} - (0.0036)^{\frac{1}{2}} = (0.0037)^{\frac{1}{2}} - 0.06 \\ \Rightarrow (0.0037)^{\frac{1}{2}} &= 0.06 + \Delta y \quad \left[\frac{1}{2} \text{ Mark}\right]\end{aligned}$$

Step 4:

Since, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right) \Delta x$$

$$\Delta y = \frac{1}{2\sqrt{x}} (\Delta x) \quad \left[\text{as } y = x^{\frac{1}{2}} \right]$$

$$= \frac{1}{2 \times 0.06} (0.0001)$$

$$= \frac{0.0001}{0.12} = 0.00083 \quad \left[\frac{1}{2} \text{ Mark} \right]$$

Step 5:

Now, by substituting the value of Δy we get:

$$(0.0037)^{\frac{1}{2}} = 0.06 + \Delta y$$

$$(0.0037)^{\frac{1}{2}} = 0.06 + 0.00083$$

$$(0.0037)^{\frac{1}{2}} = 0.06083 \quad \left[\frac{1}{2} \text{ Mark} \right]$$

Therefore, the approximate value of $(0.0037)^{\frac{1}{2}}$ is 0.06083.

(xii) **Step 1:**

Given: $(26.57)^{\frac{1}{3}}$

Step 2:

Consider $y = x^{\frac{1}{3}}$.

Let $x = 27$ and $\Delta x = -0.43$. $\left[\frac{1}{2} \text{ Mark} \right]$

Step 3:

Then,

$$\Delta y = (x + \Delta x)^{\frac{1}{3}} - (x)^{\frac{1}{3}} = (26.57)^{\frac{1}{3}} - (27)^{\frac{1}{3}} = (26.57)^{\frac{1}{3}} - 3$$

$$\Rightarrow (26.57)^{\frac{1}{3}} = 3 + \Delta y \quad \left[\frac{1}{2} \text{ Mark} \right]$$

Step 4:

Since, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right) \Delta x$$

$$\begin{aligned}\Delta y &= \frac{1}{3(x)^{\frac{2}{3}}}(\Delta x) \quad \left[\text{as } y = x^{\frac{1}{3}} \right] \\ &= \frac{1}{3(9)}(-0.43) \\ &= \frac{-0.43}{27} = -0.015 \quad \left[\frac{1}{2} \text{ Mark} \right]\end{aligned}$$

Step 5:

Now, by substituting the value of Δy we get:

$$\begin{aligned}(26.57)^{\frac{1}{3}} &= 3 + \Delta y \\ (26.57)^{\frac{1}{3}} &= 3 + (-0.015) \\ (26.57)^{\frac{1}{3}} &= 2.984 \quad \left[\frac{1}{2} \text{ Mark} \right]\end{aligned}$$

Therefore, the approximate value of $(26.57)^{\frac{1}{3}}$ is 2.984.

(xiii)**Step 1:**

Given: $(81.5)^{\frac{1}{4}}$

Step 2:

Consider $y = x^{\frac{1}{4}}$.

Let $x = 81$ and $\Delta x = 0.5$. $\left[\frac{1}{2} \text{ Mark} \right]$

Step 3:

Then,

$$\begin{aligned}\Delta y &= (x + \Delta x)^{\frac{1}{4}} - (x)^{\frac{1}{4}} \\ &= (81.5)^{\frac{1}{4}} - (81)^{\frac{1}{4}} = (81.5)^{\frac{1}{4}} - 3 \\ \Rightarrow (81.5)^{\frac{1}{4}} &= 3 + \Delta y \quad \left[\frac{1}{2} \text{ Mark} \right]\end{aligned}$$

Step 4:

Since, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx} \right) \Delta x$$

$$\Delta y = \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \quad \left[\text{as } y = x^{\frac{1}{4}} \right]$$

$$= \frac{1}{4(3)^3} (0.5) = \frac{0.5}{108} = 0.0046 \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 5:

Now, by substituting the value of Δy we get:

$$\therefore (81.5)^{\frac{1}{4}} = 3 + \Delta y$$

$$(81.5)^{\frac{1}{4}} = 3 + 0.0046$$

$$(81.5)^{\frac{1}{4}} = 3.0046 \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Hence, the approximate value of $(81.5)^{\frac{1}{4}}$ is 3.0046.

(xiv) **Step 1:**

Given: $(3.968)^{\frac{3}{2}}$

Step 2:

Consider $y = x^{\frac{3}{2}}$.

Let $x = 4$ and $\Delta x = -0.032$. $\left[\frac{1}{2} \text{ Mark}\right]$

Step 3:

Then,

$$\begin{aligned} \Delta y &= (x + \Delta x)^{\frac{3}{2}} - x^{\frac{3}{2}} \\ &= (3.968)^{\frac{3}{2}} - (4)^{\frac{3}{2}} = (3.968)^{\frac{3}{2}} - 8 \\ \Rightarrow (3.968)^{\frac{3}{2}} &= 8 + \Delta y \quad \left[\frac{1}{2} \text{ Mark}\right] \end{aligned}$$

Step 4:

Since, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right) \Delta x$$

$$\Delta y = \frac{3}{2}(x)^{\frac{1}{2}}(\Delta x) \quad \left[\text{as } y = x^{\frac{3}{2}}\right]$$

$$= \frac{3}{2}(2)(-0.032)$$

$$= -0.096 \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 5:

Now, by substituting the value of Δy we get:

$$(3.968)^{\frac{3}{2}} = 8 + \Delta y$$

$$(3.968)^{\frac{3}{2}} = 8 + (-0.096)$$

$$(3.968)^{\frac{3}{2}} = 7.904 \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Therefore, the approximate value of $(3.968)^{\frac{3}{2}}$ is 7.904.

(xv) **Step 1:**

Given: $(32.15)^{\frac{1}{5}}$

Step 2:

Consider $y = x^{\frac{1}{5}}$.

Let $x = 32$ and $\Delta x = 0.15$. $\left[\frac{1}{2} \text{ Mark}\right]$

Step 3:

Then,

$$\Delta y = (x + \Delta x)^{\frac{1}{5}} - x^{\frac{1}{5}} = (32.15)^{\frac{1}{5}} - (32)^{\frac{1}{5}} = (32.15)^{\frac{1}{5}} - 2$$

$$\Rightarrow (32.15)^{\frac{1}{5}} = 2 + \Delta y \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 4:

Since, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right) \Delta x$$

$$\Delta y = \frac{1}{5(x)^{\frac{4}{5}}} \cdot (\Delta x) \quad \left[\text{as } y = x^{\frac{1}{5}} \right]$$

$$= \frac{1}{5 \times (2)^4} (0.15)$$

$$= \frac{0.15}{80} = 0.00187 \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 5:

Now, by substituting the value of Δy we get:

$$\therefore (32.15)^{\frac{1}{5}} = 2 + \Delta y$$

$$(32.15)^{\frac{1}{5}} = 2 + 0.00187$$

$$(32.15)^{\frac{1}{5}} = 2.00187 \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Hence, the approximate value of $(32.15)^{\frac{1}{5}}$ is 2.00187.

2. Find the approximate value of $f(2.01)$, where $f(x) = 4x^2 + 5x + 2$. [2 Marks]

Solution:

Step 1:

Given:

$$f(x) = 4x^2 + 5x + 2$$

Step 2:

Let $x = 2$ and $\Delta x = 0.01$.

Then, we have:

$$\begin{aligned} f(2.01) &= f(x + \Delta x) \\ &= 4(x + \Delta x)^2 + 5(x + \Delta x) + 2 \quad \left[\frac{1}{2} \text{ Mark}\right] \end{aligned}$$

Step 3:

$$\text{Now, } \Delta y = f(x + \Delta x) - f(x)$$

$$\therefore f(x + \Delta x) = f(x) + \Delta y$$

$$= f(x) + f'(x) \cdot \Delta x \quad (\text{as } dx = \Delta x) \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 4:

$$\begin{aligned} \Rightarrow f(2.01) &= (4x^2 + 5x + 2) + (8x + 5)\Delta x \\ &= [4(2)^2 + 5(2) + 2] + [8(2) + 5](0.01) \quad [\text{as } x = 2, \Delta x = 0.01] \\ &= (16 + 10 + 2) + (16 + 5)(0.01) \\ &= 28 + (21)(0.01) \\ &= 28 + 0.21 \\ &= 28.21 \quad [1 \text{ Mark}] \end{aligned}$$

Hence, the approximate value of $f(2.01)$ is 28.21.

3. Find the approximate value of $f(5.001)$, where $f(x) = x^3 - 7x^2 + 15$ [2 Marks]

Solution:

Step 1:

Given: $f(x) = x^3 - 7x^2 + 15$

Step 2:

Let $x = 5$ and $\Delta x = 0.001$. Then, we have:

$$f(5.001) = f(x + \Delta x) = (x + \Delta x)^2 - 7(x + \Delta x)^2 + 15 \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 3:

Now, $\Delta y = f(x + \Delta x) - f(x)$

$$\therefore f(x + \Delta x) = f(x) + \Delta y$$

$$= f(x) + f'(x) \cdot \Delta x \quad (\text{as } dx = \Delta x) \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 4:

$$\begin{aligned} \Rightarrow f(5.001) &= (x^3 - 7x^2 + 15) + (3x^2 - 14x)\Delta x \\ &= [(5)^3 - 7(5)^2 + 15] + [3(5)^2 - 14(5)](0.001) \quad [x = 5, \Delta x = 0.001] \\ &= (125 - 175 + 15) + (75 - 70)(0.001) \\ &= -35 + (5)(0.001) \\ &= -35 + 0.005 \\ &= -34.995 \quad [1 \text{ Mark}] \end{aligned}$$

Hence, the approximate value of $f(5.001)$ is -34.995 .

4. Find the approximate change in the volume V of a cube of side x metres caused by increasing the side by 1%. [2 Marks]

Solution:

Step 1:

Given:

Increase in the side = 1% = $0.01x$

$$\therefore \Delta x = 0.01x$$

Step 2:

The volume of a cube (V) of side x is given by $V = x^3$.

$$dV = \left(\frac{dV}{dx}\right) \Delta x$$

$$dV = \left(\frac{d(x^3)}{dx}\right) \Delta x \quad [1 \text{ Mark}]$$

Step 3:

$$= (3x^2) \Delta x$$

$$= (3x^2)(0.01x) \quad [\text{as } 1\% \text{ of } x \text{ is } 0.01x]$$

$$= 0.03x^3 \quad [1 \text{ Mark}]$$

Therefore, the approximate change in the volume of the cube is $0.03x^3 \text{ m}^3$.

5. Find the approximate change in the surface area of a cube of side x metres caused by decreasing the side by 1%. [2 Marks]

Solution:**Step 1:****Given:**

$$\text{Decrease in the side} = 1\% = -0.01x$$

$$\therefore \Delta x = -0.01x$$

Step 2:

The surface area of a cube (S) of side x is given by $S = 6x^2$.

$$\therefore \frac{dS}{dx} = \left(\frac{dS}{dx}\right) \Delta x \quad [1 \text{ Mark}]$$

Step 3:

$$= (12x)\Delta x$$

$$= (12x)(-0.01x) \quad [\text{as } 1\% \text{ of } x \text{ is } 0.01x]$$

$$= -0.12x^2 \quad [1 \text{ Mark}]$$

Hence, the approximate change in the surface area of the cube is $-0.12x^2 \text{ m}^2$.

6. If the radius of a sphere is measured as 7 m with an error of 0.02m, then find the approximate error in calculating its volume. [2 Marks]

Solution:**Step 1:****Given:**

$$\text{Radius of a sphere} = 7 \text{ m}$$

$$\text{Error} = 0.02 \text{ m}$$

Step 2:

Let r be the radius of the sphere and Δr be the error in measuring the radius.

Then,

$$r = 7 \text{ m and } \Delta r = 0.02 \text{ m}$$

Now, the volume V of the sphere is given by,

$$V = \frac{4}{3}\pi r^3$$

$$\therefore \frac{dV}{dr} = 4\pi r^2 \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 3:

$$\therefore \Delta V = \left(\frac{dV}{dr}\right) \Delta r$$

$$= (4\pi r^2) \Delta r \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 4:

$$= 4\pi(7)^2(0.02) \text{ m}^3$$

$$\Delta V = 3.92\pi \text{ m}^3 \quad [1 \text{ Mark}]$$

Hence, the approximate error in calculating the volume is $3.92\pi \text{ m}^3$.

7. If the radius of a sphere is measured as 9 m with an error of 0.03 m, then find the approximate error in calculating in surface area. [2 Marks]

Solution:

Step 1:

Given:

Radius of a sphere = 9 m

Error = 0.03 m

Step 2:

Let r be the radius of the sphere and Δr be the error in measuring the radius.

Then,

$$r = 9 \text{ m and } \Delta r = 0.03 \text{ m}$$

Now, the surface area of the sphere (S) is given by,

$$S = 4\pi r^2$$

$$\therefore \frac{dS}{dr} = 8\pi r \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 3:

$$\begin{aligned}\therefore dS &= \left(\frac{dS}{dr}\right) \Delta r \\ &= (8\pi r) \Delta r \quad \left[\frac{1}{2} \text{ Mark}\right]\end{aligned}$$

Step 4:

$$\begin{aligned}&= 8\pi(9)(0.03) \text{ m}^2 \\ &= 2.16\pi \text{ m}^2 \quad [1 \text{ Mark}]\end{aligned}$$

Hence, the approximate error in calculating the surface area is $2.16\pi \text{ m}^2$.

8. If $f(x) = 3x^2 + 15x + 5$, then the approximate value of $f(3.02)$ is [2 Marks]

- A. 47.66
- B. 57.66
- C. 67.66
- D. 77.66

Solution:

Step 1:

Given:

$$f(x) = 3x^2 + 15x + 5$$

Step 2:

Let $x = 3$ and $\Delta x = 0.02$.

Then, we have:

$$f(3.02) = f(x + \Delta x) = 3(x + \Delta x)^2 + 15(x + \Delta x) + 5 \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 3:

$$\text{Now, } \Delta y = f(x + \Delta x) - f(x)$$

$$\Rightarrow f(x + \Delta x) = f(x) + \Delta y$$

$$= f(x) + f'(x)\Delta x \quad (\text{As } dx = \Delta x) \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 4:

$$\Rightarrow f(3.02) = (3x^2 + 15x + 5) + (6x + 15)\Delta x$$

$$= [3(3)^2 + 15(3) + 5] + [6(3) + 15](0.02) \quad [\text{As } x = 3, \Delta x = 0.02]$$

$$= (27 + 45 + 5) + (18 + 15)(0.02)$$

$$= 77 + (33)(0.02)$$

$$= 77 + 0.66$$

$$= 77.66 \quad \text{[1 Mark]}$$

Therefore, the approximate value of $f(3.02)$ is 77.66.

Hence, the correct answer is D.

9. The approximate change in the volume of a cube of side x metres caused by increasing the side by 3% is **[2 Marks]**

A. $0.06x^3m^3$

B. $0.6x^3m^3$

C. $0.09x^3m^3$

D. $0.9x^3m^3$

Solution:

Step 1:

Given:

$$\text{Increase in the side} = 3\% = 0.03x$$

$$\therefore \Delta x = 0.03x$$

Step 2:

The volume of a cube (V) of side x is given by $V = x^3$.

$$\frac{dV}{dx} = 3x^2 \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 3:

$$\therefore dV = \left(\frac{dV}{dx}\right) \Delta x$$

$$= (3x^2)\Delta x \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 4:

$$= (3x^2)(0.03x) \quad [\text{As } \% \text{ of } x \text{ is } 0.03x]$$

$$= 0.09x^3m^3$$

Therefore, the approximate change in the volume of the cube is $0.09x^3m^3$.

Hence, the correct answer is C. **[1 Mark]**

Exercise 6.5

1. Find the maximum and minimum values, if any, of the following functions given by

(i) $f(x) = (2x - 1)^2 + 3$ [2 Marks]

(ii) $f(x) = 9x^2 + 12x + 2$ [2 Marks]

(iii) $f(x) = -(x - 1)^2 + 10$ [2 Marks]

(iv) $g(x) = x^3 + 1$ [2 marks]

Solution:

(i) Step 1:

The given function is $f(x) = (2x - 1)^2 + 3$.

Step 2:

It can be observed that $(2x - 1)^2 \geq 0$ for every $x \in R$. [$\frac{1}{2}$ Mark]

Step 3:

Therefore,

$$f(x) = (2x - 1)^2 + 3 \geq 3 \text{ for every } x \in R. \quad [\frac{1}{2} \text{ Mark}]$$

Step 4:

The minimum value of f is attained when $2x - 1 = 0$.

$$2x - 1 = 0$$

$$\therefore x = \frac{1}{2} \quad [\frac{1}{2} \text{ Mark}]$$

Step 5:

Hence, the minimum value of $f = f\left(\frac{1}{2}\right) = \left(2 \cdot \frac{1}{2} - 1\right)^2 + 3 = 3$

Therefore, function f does not have a maximum value. [$\frac{1}{2}$ Mark]

(ii) Step 1:

The given function is $f(x) = 9x^2 + 12x + 2$

Step 2:

The given $f(x)$ can also be written as:

$$f(x) = (3x + 2)^2 - 2$$

It can be observed that $(3x + 2)^2 \geq 0$ for every $x \in R$. [$\frac{1}{2}$ Mark]

Step 3:

Therefore, $f(x) = (3x + 2)^2 - 2 \geq -2$ for every $x \in R$. [$\frac{1}{2}$ Mark]

Step 4:

The minimum value of f is attained when $3x + 2 = 0$.

$$3x + 2 = 0$$

$$\therefore x = \frac{-2}{3} \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 5:

Hence, the minimum value of $f = f\left(-\frac{2}{3}\right)$

$$= \left(3\left(-\frac{2}{3}\right) + 2\right)^2 - 2 = -2$$

Therefore, function f does not have a maximum value. $\left[\frac{1}{2} \text{ Mark}\right]$

(iii) Step 1;

The given function is $f(x) = -(x - 1)^2 + 10$.

Step 2:

It can be observed that $(x - 1)^2 \geq 0$ for every $x \in R$. $\left[\frac{1}{2} \text{ Mark}\right]$

Step 3:

Therefore, $f(x) = -(x - 1)^2 + 10 \leq 10$ for every $x \in R$. $\left[\frac{1}{2} \text{ Mark}\right]$

Step 4:

The maximum value of f is attained when $(x - 1) = 0$.

$$(x - 1) = 0$$

$$\therefore x = 0 \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 5:

Hence, the maximum value of $f = f(1) = -(1 - 1)^2 + 10 = 10$

Therefore, function f does not have a minimum value. $\left[\frac{1}{2} \text{ Mark}\right]$

(iv) Step 1:

The given function is $g(x) = x^3 + 1$.

Step 2:

$$g'(x) = \frac{d(x^3+1)}{dx} = 3x^2 \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 3:

Putting $g'(x) = 0$

$$3x^2 = 0$$

$$x^2 = 0$$

$$x = 0 \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 4:

Hence by the first derivative test,

The point $x = 0$ is neither a point of local maxima nor a point of local minima

Hence, $x = 0$ is point of inflexion.

Therefore, there is no maximum value or a minimum value. **[1 Mark]**

2. Find the maximum and minimum values, if any, of the following functions given by

(i) $f(x) = |x + 2| - 1$ **[2 Marks]**

(ii) $g(x) = -|x + 1| + 3$ **[2 Marks]**

(iii) $h(x) = \sin(2x) + 5$ **[2 Marks]**

(iv) $f(x) = |\sin 4x + 3|$ **[2 Marks]**

(v) $h(x) = x + 1, x \in (-1, 1)$ **[2 Marks]**

Solution:

(i) **Step 1:**

Given: $f(x) = |x + 2| - 1$

Step 2:

We know that $|x + 2| \geq 0$ for every $x \in R$. **$\left[\frac{1}{2} \text{ Mark}\right]$**

Step 3:

Thus, $f(x) = |x + 2| - 1 \geq -1$ for every $x \in R$. **$\left[\frac{1}{2} \text{ Mark}\right]$**

Step 4:

The minimum value of f is attained when $|x + 2| = 0$

$$|x + 2| = 0$$

$$\Rightarrow x = -2 \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 5:

Hence, the minimum value of $f = f(-2)$

$$= |-2 + 2| - 1 = -1$$

Therefore, function f does not have a maximum value. $[\frac{1}{2} \text{ Mark}]$

(ii) **Step 1:**

Given: $g(x) = -|x + 1| + 3$

Step 2:

We know that for every $x \in R$.

$$-|x + 1| \leq 0 \quad [\frac{1}{2} \text{ Mark}]$$

Step 3:

Since, $g(x) = -|x + 1| + 3 \leq 3$ for every $x \in R$. $[\frac{1}{2} \text{ Mark}]$

Step 4:

The maximum value of g is attained when $|x + 1| = 0$

$$|x + 1| = 0$$

$$\Rightarrow x = -1 \quad [\frac{1}{2} \text{ Mark}]$$

Step 5:

Hence, the maximum value of $g = g(-1)$

$$= -|1 + 1| + 3 = 3$$

Therefore, function g does not have a minimum value. $[\frac{1}{2} \text{ Mark}]$

(iii) **Step 1:**

Given: $h(x) = \sin 2x + 5$

Step 2:

We know that $-1 \leq \sin 2x \leq 1$. $[\frac{1}{2} \text{ Mark}]$

Step 3:

Now, adding 5 on both the sides we get:

$$-1 + 5 \leq \sin 2x + 5 \leq 1 + 5 \quad [\frac{1}{2} \text{ Mark}]$$

Step 4:

$$\therefore 4 \leq \sin 2x + 5 \leq 6$$

$$4 \leq f(x) \leq 6 \quad [\frac{1}{2} \text{ Mark}]$$

Step 5:

Hence, the maximum and minimum values of h are 6 and 4 respectively. $[\frac{1}{2} \text{ Mark}]$

(iv) **Step 1:**

Given: $f(x) = |\sin 4x + 3|$

Step 2:

We know that $-1 \leq \sin 4x \leq 1$. $[\frac{1}{2} \text{ Mark}]$

Step 3:

Now, adding 3 on both the sides we get:

$$-1 + 3 \leq \sin 4x + 3 \leq 1 + 3$$

$$2 \leq \sin 4x + 3 \leq 4 \quad [\frac{1}{2} \text{ Mark}]$$

Step 4:

By taking modulus we get:

$$\therefore 2 \leq |\sin 4x + 3| \leq 4$$

$$4 \leq f(x) \leq 2 \quad [\frac{1}{2} \text{ Mark}]$$

Step 5:

Hence, the maximum and minimum values of f are 4 and 2 respectively. $[\frac{1}{2} \text{ Mark}]$

(iv) **Step 1:**

Given: $h(x) = x + 1, x \in (-1, 1)$

Step 2:

$$h(x) = x + 1,$$

$$x \in (-1, 1)$$

Meaning, $-1 < x < 1$ $[\frac{1}{2} \text{ Mark}]$

Step 3:

Adding 1 throughout:

$$0 < x + 1 < 2$$

$$0 < h(x) < 2 \quad [\frac{1}{2} \text{ Mark}]$$

Step 4:

Therefore, function $h(x)$ has neither maximum nor minimum value in $(-1, 1)$. **[1 Mark]**

3. Find the local maxima and local minima, if any, of the following functions. Find also the local maximum and the local minimum values, as the case may be:

(i) $f(x) = x^2$ [2 Marks]

(ii) $g(x) = x^3 - 3x$ [2 Marks]

(iii) $h(x) = \sin x + \cos x, 0 < x < \frac{\pi}{2}$ [2 Marks]

(iv) $f(x) = \sin x - \cos x, 0 < x < 2\pi$ [4 Marks]

(v) $f(x) = x^3 - 6x^2 + 9x + 15$ [4 Marks]

(vi) $g(x) = \frac{x}{2} + \frac{2}{x}, x > 0$ [4 Marks]

(vii) $g(x) = \frac{1}{x^2+2}$ [2 Marks]

(viii) $f(x) = x\sqrt{1-x}, x > 0$ [4 marks]

Solution:

(i) **Step 1:**

Given:

$$f(x) = x^2$$

Step 2:

$$\therefore f'(x) = 2x$$

Now, let us equate it to zero:

$$f'(x) = 0$$

$$\Rightarrow x = 0 \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 3:

Thus, $x = 0$ is the only critical point which could possibly be the point of local maxima or local minima of f . $\left[\frac{1}{2} \text{ Mark}\right]$

Step 4:

We have, $f''(0) = 2$ which is positive.

Therefore, by second derivative test, $x = 0$ is a point of local minima. $\left[\frac{1}{2} \text{ Mark}\right]$

Step 5:

Local minimum value of f at $x = 0$ is $f(0) = 0$. $\left[\frac{1}{2} \text{ Mark}\right]$

(ii) **Step 1:**

Given: $g(x) = x^3 - 3x$

Step 2:

$$\therefore g'(x) = 3x^2 - 3$$

Now,

$$g'(x) = 0$$

$$\Rightarrow 3x^2 = 3$$

$$\Rightarrow x = \pm 1 \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 3:

$$g'(x) = 6x$$

$$g'(1) = 6 > 0$$

$$g'(-1) = -6 < 0 \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 4:

By second derivative test,

$x = 1$ is a point of local minima and local minimum value of g at $x = 1$ is $g(1) = 1^3 - 3 = 1 - 3 = -2$. $\left[\frac{1}{2} \text{ Mark}\right]$

Step 5:

Also, $x = -1$ is a point of local maxima and local maximum value of g at $x = -1$ is $g(-1) = (-1)^3 - 3(-1) = -1 + 3 = 2$. $\left[\frac{1}{2} \text{ Mark}\right]$

(iii) **Step 1:**

Given: $h(x) = \sin x + \cos x, 0 < x < \frac{\pi}{2}$

Step 2:

$$\therefore h'(x) = \cos x - \sin x$$

$$h'(x) = 0$$

$$\Rightarrow \sin x = \cos x$$

$$\Rightarrow \tan x = 1$$

$$\Rightarrow x = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right) \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 3:

$$h''(x) = -\sin x - \cos x$$

$$= -(\sin x + \cos x)$$

$$h''\left(\frac{\pi}{4}\right) = -\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$= -\frac{2}{\sqrt{2}} = -\sqrt{2} < 0 \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 4:

Therefore, by second derivative test, $x = \frac{\pi}{4}$ is a point of local maxima and the local maximum value of h at $x = \frac{\pi}{4}$ is

$$h\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}. \quad [1 \text{ Mark}]$$

(iv)**Step 1:**

Given: $f(x) = \sin x - \cos x, 0 < x < 2\pi$

Step 2:

$$\therefore f'(x) = \cos x + \sin x$$

$$f'(x) = 0$$

$$\Rightarrow \cos x = -\sin x$$

$$\Rightarrow \tan x = -1$$

$$\Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4} \in (0, 2\pi) \quad [1 \text{ Mark}]$$

Step 3:

$$f''(x) = -\sin x + \cos x$$

$$f''\left(\frac{3\pi}{4}\right) = -\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2} < 0$$

$$f''\left(\frac{7\pi}{4}\right) = -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} > 0 \quad [1 \text{ Mark}]$$

Step 4:

Hence, by second derivative test, $x = \frac{3\pi}{4}$ is a point of local maxima and the local maximum value of f at $x = \frac{3\pi}{4}$ is $f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$ [1 mark]

Step 5:

Also, $x = \frac{7\pi}{4}$ is a point of local minima and the local minimum value of f is

$$f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2} \quad [1 \text{ Mark}]$$

(v)**Step 1:**

Given: $f(x) = x^3 - 6x^2 + 9x + 15$

Step 2:

$$\therefore f'(x) = 3x^2 - 12x + 9 \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 3:

$$f'(x) = 0 \Rightarrow 3(x^2 - 4x + 3) = 0$$

$$\Rightarrow 3(x - 1)(x - 3) = 0$$

$$\Rightarrow x = 1, 3 \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 4:

$$\text{Now, } f''(x) = 6x - 12 = 6(x - 2)$$

$$f''(1) = 6(1 - 2) = -6 < 0$$

$$f''(3) = 6(3 - 2) = 6 > 0 \quad [1 \text{ Mark}]$$

Step 5:

Hence, by second derivative test, $x = 1$ is a point of local maxima and the local maximum value of f at $x = 1$ is

$$f(1) = 1 - 6 + 9 + 15 = 19. \quad [1 \text{ Mark}]$$

Step 6:

However, $x = 3$ is a point of local minima and the local minimum value of f at $x = 3$ is

$$f(3) = 27 - 54 + 27 + 15 = 15. \quad [1 \text{ Mark}]$$

(vi) **Step 1:**

Given: $g(x) = \frac{x}{2} + \frac{2}{x}, x > 0$

Step 2:

$$\therefore g'(x) = \frac{1}{2} - \frac{2}{x^2} \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 3:

Now,

$$g'(x) = 0 \text{ Gives } \frac{2}{x^2} = \frac{1}{2}$$

$$\Rightarrow x^2 = 4$$

$$\Rightarrow x = \pm 2 \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 4:

Since $x > 0$, we take $x = 2$.

Now,

$$g''(x) = \frac{4}{x^3}$$

$$g''(2) = \frac{4}{2^3} = \frac{1}{2} > 0 \quad \text{[1 Mark]}$$

Step 5:

Hence, by second derivative test,

$x = 2$ is a point of local minima [1 Mark]

Step 6:

The local minimum value of g at $x = 2$ is

$$g(2) = \frac{2}{2} + \frac{2}{2} = 1 + 1 = 2. \quad \text{[1 mark]}$$

(vii) **Step 1:**

$$\text{Given: } g(x) = \frac{1}{x^2+2}$$

Step 2:

$$\therefore g'(x) = \frac{-(2x)}{(x^2+2)^2} \quad \text{[1/2 Mark]}$$

Step 3:

$$g'(x) = 0 \Rightarrow \frac{-2x}{(x^2+2)^2} = 0 \Rightarrow x = 0 \quad \text{[1/2 Mark]}$$

Step 4:

$$g''(x) = \frac{-2(2-3x^2)}{(x^2+2)^3}$$

Therefore, at $x = 0$

$$g''(x) = \frac{-2(2-3(0)^2)}{((0)^2+2)^3} = -\frac{1}{2} < 0 \quad \text{[1/2 Mark]}$$

Step 5:

Hence, by second derivative test,

$x = 0$ is a point of local maxima and the local maximum value of $g(0)$ is $\frac{1}{0+2} = \frac{1}{2}$. [1/2 Mark]

(viii) **Step 1:**

$$\text{Given: } f(x) = x\sqrt{1-x}, x > 0$$

Step 2:

$$\begin{aligned} \therefore f'(x) &= \sqrt{1-x} + x \cdot \frac{1}{2\sqrt{1-x}}(-1) \\ &= \sqrt{1-x} - \frac{x}{2\sqrt{1-x}} \\ &= \frac{2(1-x) - x}{2\sqrt{1-x}} \\ &= \frac{2-3x}{2\sqrt{1-x}} \quad \text{[1 Mark]} \end{aligned}$$

Step 3:

Now,

$$\begin{aligned} f'(x) &= 0 \\ \Rightarrow \frac{2-3x}{2\sqrt{1-x}} &= 0 \\ \Rightarrow 2-3x &= 0 \\ \Rightarrow x &= \frac{2}{3} \quad \text{[1/2 Mark]} \end{aligned}$$

Step 4:

$$\begin{aligned} \text{So, } f''(x) &= \frac{1}{2} \left[\frac{\sqrt{1-x}(-3) - (2-3x)\left(\frac{-1}{2\sqrt{1-x}}\right)}{1-x} \right] \\ &= \frac{\sqrt{1-x}(-3) + (2-3x)\left(\frac{1}{2\sqrt{1-x}}\right)}{2(1-x)} \\ &= \frac{-6(1-x) + (2-3x)}{4(1-x)^{\frac{3}{2}}} \\ &= \frac{3x-4}{4(1-x)^{\frac{3}{2}}} \quad \text{[1 mark]} \end{aligned}$$

Using quotient rule $\left[\left(\frac{u}{v}\right)'\right] = \frac{u'v - v'u}{v^2}$

Step 5:

Now, $x = \frac{2}{3}$

$$\begin{aligned} f''\left(\frac{2}{3}\right) &= \frac{3\left(\frac{2}{3}\right) - 4}{4\left(1 - \frac{2}{3}\right)^{\frac{3}{2}}} \\ &= \frac{2-4}{4\left(\frac{1}{3}\right)^{\frac{3}{2}}} = \frac{-1}{2\left(\frac{1}{3}\right)^{\frac{3}{2}}} < 0 \quad \text{[1/2 Mark]} \end{aligned}$$

Step 6:

Hence, by second derivative test, $x = \frac{2}{3}$ is a point of local maxima and the local maximum value of f at $x = \frac{2}{3}$ is

$$f\left(\frac{2}{3}\right) = \frac{2}{3} \sqrt{1 - \frac{2}{3}} = \frac{2}{3} \sqrt{\frac{1}{3}} = \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}}{9}. \quad [1 \text{ Mark}]$$

4. Prove that the following functions do not have maxima or minima:

(i). $f(x) = e^x$ [2 Marks]

(ii) $g(x) = \log x$ [2 marks]

(iii). $h(x) = x^3 + x^2 + x + 1$ [2 Marks]

Solution:

(i) **Step 1:**

Given:

$$f(x) = e^x$$

Step 2:

$$\text{Now, if } f'(x) = e^x = 0,$$

$$\text{Then } e^x = 0$$

But the exponential function can never assume the value 0 for any value of x . [1 Mark]

Step 3:

Hence, there does not exist $c \in R$ such that $f'(c) = 0$.

Therefore, function f does not have maxima or minima. [1 Mark]

Hence proved.

(ii) **Step 1:**

Given:

$$g(x) = \log x$$

Step 2:

$$\therefore g'(x) = \frac{1}{x}$$

Since $\log x$ is defined for a positive number x , $g'(x) > 0$ for any x . [1 Mark]

Step 3:

Therefore, there does not exist $c \in R$ such that $g'(c) = 0$

Therefore, function g does not have maxima or minima. [1 Mark]

Hence proved.

(iii) **Step 1:**

Given:

$$h(x) = x^3 + x^2 + x + 1$$

Step 2:

$$\therefore h'(x) = 3x^2 + 2x + 1$$

Now,

$$h(x) = 0$$

$$\Rightarrow 3x^2 + 2x + 1 = 0 \text{ [1 Mark]}$$

Step 3:

$$\Rightarrow x = \frac{-2 \pm 2\sqrt{2}i}{6} = \frac{-1 \pm \sqrt{2}i}{3} \notin \mathbb{R}$$

Hence, there does not exist $c \in \mathbb{R}$ such that $h'(c) = 0$.

Therefore, function h does not have maxima or minima. [1 Mark]

Hence proved.

5. Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals:

(i) $f(x) = x^3, x \in [-2, 2]$ [2 Marks]

(ii) $f(x) = \sin x + \cos x, x \in [0, \pi]$ [4 marks]

(iii) $f(x) = 4x - \frac{1}{2}x^2, x \in \left[-2, \frac{9}{2}\right]$ [4 Marks]

(iv) $f(x) = (x - 1)^2 + 3, x \in [-3, 1]$ [4 Marks]

Solution:

(i) **Step 1:**

The given function is $f(x) = x^3$.

Step 2:

$$\therefore f'(x) = 3x^2$$

Now,

$$f'(x) = 0$$

$$\Rightarrow x = 0$$

Then, we evaluate the value of f at critical point $x = 0$ and at end points of the interval $[-2, 2]$. [$\frac{1}{2}$ Mark]

Step 3:

$$f(0) = 0$$

$$f(-2) = (-2)^3 = -8$$

$$f(2) = (2)^3 = 8 \text{ } [\frac{1}{2} \text{ Mark}]$$

Step 4:

Therefore, we can conclude that the absolute maximum value of f on $[-2, 2]$ is 8 occurring at $x = 2$. Also, the absolute minimum value of f on $[-2, 2]$ is -8 occurring at $x = -2$. [$\frac{1}{2}$ Mark]

Step 5:

Therefore, the required absolute maximum value and the absolute minimum value for the given function are 8 and -8 respectively. [$\frac{1}{2}$ Mark]

(ii) **Step 1:**

The given function is $f(x) = \sin x + \cos x$.

Step 2:

$$\therefore f'(x) = \cos x - \sin x$$

Now,

$$f'(x) = 0$$

$$\Rightarrow \sin x = \cos x$$

$$\Rightarrow \tan x = 1$$

$$\Rightarrow x = \frac{\pi}{4}$$

Then, we evaluate the value of f at critical point $x = \frac{\pi}{4}$ and at the end points of the interval $[0, \pi]$. [1 Mark]

Step 3:

$$f\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{4} + \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$f(0) = \sin 0 + \cos 0 = 0 + 1 = 1$$

$$f(\pi) = \sin \pi + \cos \pi = 0 - 1 = -1 \text{ } [1\frac{1}{2} \text{ Mark}]$$

Step 4:

Hence, we can conclude that the absolute maximum value of f on $[0, \pi]$ is $\sqrt{2}$ occurring at $x = \frac{\pi}{4}$. Also, the absolute minimum value of f on $[0, \pi]$ is -1 occurring at $x = \pi$.

Therefore, the required absolute maximum value and the absolute minimum value for the given function are $\sqrt{2}$ and -1 respectively. [$1\frac{1}{2}$ Mark]

(iii) **Step 1:**

The given function is $f(x) = 4x - \frac{1}{2}x^2$.

Step 2:

$$\therefore f'(x) = 4 - \frac{1}{2}(2x) = 4 - x$$

Now,

$$f'(x) = 0$$

$$\Rightarrow x = 4$$

Then, we evaluate the value of f at critical point $x = 4$ and at the end points of the interval $\left[-2, \frac{9}{2}\right]$ [1 Mark]

Step 3:

$$f(4) = 16 - \frac{1}{2}(16) = 16 - 8 = 8$$

$$f(-2) = -8 - \frac{1}{2}(4) = -8 - 2 = -10$$

$$f\left(\frac{9}{2}\right) = 4\left(\frac{9}{2}\right) - \frac{1}{2}\left(\frac{9}{2}\right)^2 = 18 - \frac{81}{8} = 18 - 10.125 = 7.875 \quad [1\frac{1}{2} \text{ Mark}]$$

Step 4:

Hence, we can conclude that the absolute maximum value of f on $\left[-2, \frac{9}{2}\right]$ is 8 occurring at $x = 4$ and the absolute minimum value of f on $\left[-2, \frac{9}{2}\right]$ is -10 occurring at $x = -2$.

Therefore, the required absolute maximum value and the absolute minimum value for the given function are 8 and -10 respectively. [$1\frac{1}{2}$ Mark]

(iv) **Step 1:**

The given function is $f(x) = (x - 1)^2 + 3$

Step 2:

$$\therefore f'(x) = 2(x - 1)$$

Now,

$$f'(x) = 0$$

$$\Rightarrow 2(x - 1) = 0$$

$$\therefore x = 1$$

Then, we evaluate the value of f at critical point $x = 1$ and at the end points of the interval $[-3, 1]$. **[1 Mark]**

Step 3:

$$f(1) = (1 - 1)^2 + 3 = 0 + 3 = 3$$

$$f(-3) = (-3 - 1)^2 + 3 = 16 + 3 = 19 \quad \left[1\frac{1}{2} \text{ Mark}\right]$$

Step 4:

Hence, we can conclude that the absolute maximum value of f on $[-3, 1]$ is 19 occurring at $x = -3$ and the minimum value of f on $[-3, 1]$ is 3 occurring at $x = 1$.

Therefore, the required absolute maximum value and the absolute minimum value for the given function are 19 and 3 respectively.

6. Find the maximum profit that a company can make, if the profit function is given by $p(x) = 41 - 72x - 18x^2$ **[4 Marks]**

Solution:

Step 1:

The profit function is given as $p(x) = 41 - 72x - 18x^2$.

Step 2:

$$\therefore p'(x) = -72 - 36x$$

$$p''(x) = -36 \quad \text{[1 Mark]}$$

Step 3:

Now,

$$p'(x) = 0$$

$$\Rightarrow x = \frac{-72}{-36} = 2$$

Also,

$$p''(2) = -36 < 0 \quad \text{[1 Mark]}$$

Step 4:

By second derivative test, $x = 2$ is the point of local maxima of p .

Hence, Maximum profit = $p(2)$ **[1 Mark]**

Step 5:

$$\begin{aligned}
 &= 41 - 72(-2) - 18(-2)^2 \\
 &= 41 + 144 - 72 \\
 &= 113 \text{ [1 Mark]}
 \end{aligned}$$

Therefore, the maximum profit that the company can make is 113 units.

7. Find both the maximum value and the minimum value of $3x^4 - 8x^3 + 12x^2 - 48x + 25$ on the interval $[0, 3]$. [4 Marks]

Solution:

Step 1:

$$\text{Let } f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$$

Step 2:

$$\begin{aligned}
 \therefore f'(x) &= 12x^3 - 24x^2 + 24x - 48 \\
 &= 12(x^3 - 2x^2 + 2x - 4) \\
 &= 12[x^2(x - 2) + 2(x - 2)] \\
 &= 12(x - 2)(x^2 + 2) \text{ [}\frac{1}{2}\text{ Mark]}
 \end{aligned}$$

Step 3:

Now,

$$f'(x) = 0 \text{ gives } x = 2 \text{ or } x^2 + 2 = 0 \text{ for which there are no real roots.}$$

$$\text{Hence, we consider only } x = 2 \in [0, 3] \text{ [}\frac{1}{2}\text{ Mark]}$$

Step 4:

Now, we evaluate the value of f at critical point $x = 2$ and at the end points of the interval $[0, 3]$.

$$\begin{aligned}
 f(2) &= 3(16) - 8(8) + 12(4) - 48(2) + 25 \\
 &= 48 - 64 + 48 - 96 + 25 \\
 &= -39
 \end{aligned}$$

$$\begin{aligned}
 f(0) &= 3(0) - 8(0) + 12(0) - 48(0) + 25 \\
 &= 25
 \end{aligned}$$

$$\begin{aligned}
 f(3) &= 3(81) - 8(27) + 12(9) - 48(3) + 25 \\
 &= 240 - 216 + 108 - 144 + 25 = 16 \text{ [}\frac{1}{2}\text{ Mark]}
 \end{aligned}$$

Step 5:

Therefore, we can conclude that the absolute maximum value of f on $[0,3]$ is 25 occurring at $x = 0$ and the absolute minimum value of f at $[0,3]$ is -39 occurring at $x = 2$. **[1½ Mark]**

8. At what points in the interval $[0, 2\pi]$, does the function $\sin 2x$ attain its maximum value?
[4 Marks]

Solution:

Step 1:

Let $f(x) = \sin 2x$.

Step 2:

$\therefore f'(x) = 2 \cos 2x$

Now,

$f'(x) = 0$

$\Rightarrow \cos 2x = 0$

$\Rightarrow 2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$\Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ **[1 Mark]**

Step 3:

Then, we evaluate the values of f at critical points $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ and at the end points of the interval $[0, 2\pi]$.

$f\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{2} = 1,$

$f\left(\frac{3\pi}{4}\right) = \sin \frac{3\pi}{2} = -1$

$f\left(\frac{5\pi}{4}\right) = \sin \frac{5\pi}{2} = 1,$

$f\left(\frac{7\pi}{4}\right) = \sin \frac{7\pi}{2} = -1$

$f(0) = \sin 0 = 0, f(2\pi) = \sin 2\pi = 0$ **[2 Marks]**

Step 4:

Therefore, we can conclude that the absolute maximum value of f on $[0, 2\pi]$ is occurring at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$. **[1 Mark]**

9. What is the maximum value of the function $\sin x + \cos x$? [4 marks]

Solution:

Step 1:

$$\text{Let } f(x) = \sin x + \cos x.$$

Step 2:

Now, $f''(x)$ will be negative when $(\sin x + \cos x)$ is positive i.e., when $\sin x$ and $\cos x$ are both positive. Also, we know that $\sin x$ and $\cos x$ both are positive in the first quadrant.

Then, $f''(x)$ will be negative when $x \in \left(0, \frac{\pi}{2}\right)$. That is:

$$f'(x) = 0$$

$$\Rightarrow \cos x - \sin x = 0$$

$$\Rightarrow \tan x = 1$$

Thus, we consider $x = \frac{\pi}{4}$ [1 Mark]

Step 3:

$$f''\left(\frac{\pi}{4}\right) = -\left(\sin \frac{\pi}{4} + \cos \frac{\pi}{4}\right) = -\left(\frac{2}{\sqrt{2}}\right) = -\sqrt{2} < 0. \quad [1 \text{ Mark}]$$

Step 4:

Therefore, by second derivative test,

f will be the maximum at $x = \frac{\pi}{4}$ and the maximum value of f is

$$f\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{4} + \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

Hence, the maximum value of the given function is $\sqrt{2}$. . [2 Marks]

10. Find the maximum value of $2x^3 - 24x + 107$ in the interval $[1, 3]$. Find the maximum value of the same function in $[-3, -1]$. [4 marks]

Solution:

Step 1:

$$\text{Let } f(x) = 2x^3 - 24x + 107.$$

Step 2:

$$\therefore f'(x) = 6x^2 - 24 = 6(x^2 - 4)$$

Now,

$$f'(x) = 0$$

$$\Rightarrow 6(x^2 - 4) = 0$$

$$\Rightarrow x^2 = 4$$

$$\Rightarrow x = \pm 2 \text{ . [1 Mark]}$$

Step 3:

Let us consider the interval $[1,3]$.

Then, we evaluate the value of f at the critical point $x = 2$ and at the end points of the interval $[1,3]$.

$$f(2) = 2(8) - 24(2) + 107$$

$$= 16 - 48 + 107 = 75$$

$$f(1) = 2(1) - 24(1) + 107$$

$$= 2 - 24 + 107 = 85$$

$$f(3) = 2(27) - 24(3) + 107$$

$$= 54 - 72 + 107 = 89 \text{ . [1 Mark]}$$

Step 4:

Therefore, the absolute maximum value of $f(x)$ in the interval $[1,3]$ is 89 occurring at $x = 3$.

$[\frac{1}{2}$ Mark]

Step 5:

Next, we consider the interval $[-3, -1]$.

Evaluate the value of f at the critical point $x = -2$ and at the end points of the interval $[-3, -1]$.

$$f(-3) = 2(-27) - 24(-3) + 107$$

$$= -54 + 72 + 107 = 125$$

$$f(-1) = 2(-1) - 24(-1) + 107$$

$$= -2 + 24 + 107 = 129$$

$$f(-2) = 2(-8) - 24(-2) + 107$$

$$= -16 + 48 + 107 = 139 \text{ . [1 Mark]}$$

Step 6:

Therefore, the absolute maximum value of $f(x)$ in the interval $[-3, -1]$ is 139 occurring at $x = -2$.

$[\frac{1}{2}$ Mark]

11. It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval $[0, 2]$. Find the value of a . [2 Marks]

Solution:

Step 1:

$$\text{Let } f(x) = x^4 - 62x^2 + ax + 9.$$

Step 2:

$$\therefore f'(x) = 4x^3 - 124x + a$$

It is given that function f attains its maximum value on the interval $[0, 2]$ at $x = 1$.

$$\therefore f'(1) = 0 \text{ [1 Mark]}$$

Step 3:

$$\Rightarrow 4 - 124 + a = 0$$

$$\Rightarrow a = 120 \text{ [1 Mark]}$$

Therefore, the value of a is 120.

12. Find the maximum and minimum values of $x + \sin 2x$ on $[0, 2\pi]$ [4 Marks]

Solution:

Step 1:

$$\text{Let } f(x) = x + \sin 2x.$$

Step 2:

$$\therefore f'(x) = 1 + 2 \cos 2x$$

$$\text{Now, } f'(x) = 0$$

$$\Rightarrow \cos 2x = -\frac{1}{2} = -\cos \frac{\pi}{3}$$

$$= \cos \left(\pi - \frac{\pi}{3} \right)$$

$$\cos 2x = \cos \frac{2\pi}{3} \text{ [1/2 Mark]}$$

Step 3:

General solution for $\cos 2x$ is

$$2x = 2\pi \pm \frac{2\pi}{3}, n \in \mathbf{Z}$$

$$\Rightarrow x = n\pi \pm \frac{\pi}{3}, n \in \mathbf{Z}$$

Putting $n=0$, we get:

$$\Rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \in [0, 2\pi] \quad \left[\frac{1}{2} \text{ Mark}\right]$$

Step 4:

$$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + \sin \frac{2\pi}{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

$$f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sin \frac{4\pi}{3} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

$$f\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} + \sin \frac{8\pi}{3} = \frac{4\pi}{3} + \frac{\sqrt{3}}{2}$$

$$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sin \frac{10\pi}{3} = \frac{5\pi}{3} - \frac{\sqrt{3}}{2}$$

$$f(0) = 0 + \sin 0 = 0$$

$$f(2\pi) = 2\pi + \sin 4\pi = 2\pi + 0 = 2\pi \quad [2 \text{ Marks}]$$

Step 5:

Therefore, we can conclude that the absolute maximum value of $f(x)$ in the interval $[0, 2\pi]$ is 2π occurring at $x = 2\pi$ and the absolute minimum value of $f(x)$ in the interval $[0, 2\pi]$ is 0 occurring at $x = 0$. [1 Mark]

13. Find two numbers whose sum is 24 and whose product is as large as possible. [4 Marks]

Solution:

Step 1:

Let one number be x .

Then, the other number is $(24 - x)$.

Given: $x + (24 - x) = 24$; Product of x and $24 - x$ is maximum.

Step 2:

Let $P(x)$ denote the product of the two numbers.

So, we have

$$P(x) = x(24 - x) = 24x - x^2$$

$$\therefore P'(x) = 24 - 2x$$

$$P''(x) = -2 \quad [1 \text{ Mark}]$$

Step 3:

Now,

$$P'(x) = 0$$

$$\Rightarrow x = 12$$

Also,

$$P''(12) = -2 < 0 \text{ [1}\frac{1}{2}\text{ mark]}$$

Step 4:

Hence, by second derivative test, $x = 12$ is the point of local maxima of P .

Therefore, the product of the numbers is the maximum when the numbers are 12 and $24 - 12 = 12$. [1 $\frac{1}{2}$ Mark]

14. Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum. [4 marks]

Solution:

Step 1:

Given: The two numbers are x and y such that $x + y = 60$.

$$\therefore y = 60 - x$$

Step 2:

$$\text{Let } f(x) = xy^3$$

$$\Rightarrow f(x) = x(60 - x)^3$$

Step 3:

$$\therefore f'(x) = (60 - x)^3 - 3x(60 - x)^2$$

$$= (60 - x)^2[60 - x - 3x]$$

$$= (60 - x)^2(60 - 4x)$$

$$\text{Now, } f'(x) = 0$$

$$\Rightarrow x = 60 \text{ or } x = 15 \text{ [1 Mark]}$$

Step 4:

$$\text{And, } f''(x) = -2(60 - x)(60 - 4x) - 4(60 - x)^2$$

$$= -2(60 - x)[60 - 4x + 2(60 - x)]$$

$$= -2(60 - x)(180 - 6x) \text{ [1 Mark]}$$

Step 5:

When, $x = 60, f''(x) = 0$

When, $x = 15, f''(x) = -12(60 - 15)(180 - 90) = -8100 < 0$

Hence, by second derivative test, $x = 15$ is a point of local maxima of f . [1 Mark]

Step 6:

Thus, function xy^3 is maximum when $x = 15$ and $y = 60 - 15 = 45$.

Therefore, the required numbers are 15 and 45. [1 Mark]

15. Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is a maximum

[6 Marks]

Solution:

Step 1:

Given: Two positive numbers x and y are such that their sum is 35 and the product x^2y^5 is a maximum

Let one number be x .

Then, the other number is $y = (35 - x)$.

Let $P(x) = x^2y^5$.

Step 2:

Then, we have

$$P(x) = x^2(35 - x)^5$$

$$\therefore P'(x) = 2x(35 - x)^5 - 5x^2(35 - x)^4$$

$$= x(35 - x)^4[2(35 - x) - 5x]$$

$$= x(35 - x)^4(70 - 7x)$$

$$= 7x(35 - x)^4(10 - x)$$

Now, $P'(x) = 0 \Rightarrow x = 0, 35, 10$ [1 $\frac{1}{2}$ Marks]

Step 3:

$$\text{And, } P''(x) = 7(35 - x)^4(10 - x) + 7x[-(35 - x)^4 - 4(35 - x)^3(10 - x)]$$

$$= 7(35 - x)^4(10 - x) - 7x(35 - x)^4 - 28x(35 - x)^3(10 - x)$$

$$= 7(35 - x)^3[(35 - x)(10 - x) - x(35 - x) - 4x(10 - x)]$$

$$= 7(35 - x)^3 [350 - 45x + x^2 - 35x + x^2 - 40x + 4x^2]$$

$$= 7(35 - x)^3 (6x^2 - 120x + 350) \left[1\frac{1}{2} \text{ Marks}\right]$$

Step 4:

When $x = 35, y = 35 - 35 = 0$ and the product x^2y^5 will be equal to 0.

When $x = 0, y = 35 - 0 = 35$ and the product x^2y^5 will be 0.

$\therefore x = 0$ and $x = 35$ cannot be the possible values of x .

When $x = 10$, we have

$$P''(x) = 7(35 - 10)^3 (6 \times 100 - 120 \times 10 + 350)$$

$$= 7(25)^3 (-250) < 0 \left[1\frac{1}{2} \text{ Marks}\right]$$

Step 5:

Hence, by second derivative test, $P(x)$ will be the maximum when $x = 10$ and $y = 35 - 10 = 25$.

Therefore, the required numbers are 10 and 25. $\left[1\frac{1}{2} \text{ Marks}\right]$

16. Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.

[4 Marks]

Solution:

Step 1:

Given: Two positive numbers are such that their sum is 16 and the sum of their cubes is minimum.

Let one number be x .

Then, the other number is $(16 - x)$.

Let the sum of the cubes of these numbers be denoted by $S(x)$.

Step 2:

Then,

$$S(x) = x^3 + (16 - x)^3$$

$$\therefore S'(x) = 3x^2 - 3(16 - x)^2$$

$$S''(x) = 6x + 6(16 - x) \left[1 \text{ Mark}\right]$$

Step 3:

Now, $S'(x) = 0$

$\Rightarrow 3x^2 - 3(16 - x)^2 = 0$

$\Rightarrow x^2 - (16 - x)^2 = 0$

$\Rightarrow x^2 - 256 - x^2 + 32x = 0$

$\Rightarrow x = \frac{256}{32} = 8$ [1 Mark]

Step 4:

So, $S''(8) = 6(8) + 6(16 - 8)$

$= 48 + 48 = 96 > 0$ [1 Mark]

Step 5:

Hence, by second derivative test, $x = 8$ is the point of local minima of S .

Therefore, the sum of the cubes of the numbers is the minimum when the numbers are 8 and $16 - 8 = 8$. [1 Mark]

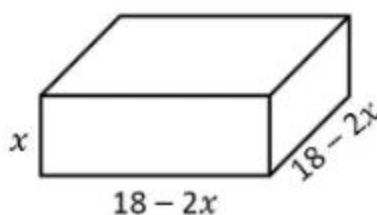
17. A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible? [4 Marks]

Solution:

Step 1:

Given:

A square piece of tin has side 18cm.



Let the side of the square to be cut off be x cm.

Step 2:

Length after removing = $18 - x - x = 18 - 2x$

Breadth after removing = $18 - x - x = 18 - 2x$

Height of the box = x [1 Mark]

Step 3:

Hence, the volume $V(x)$ of the box is given by,

$$V(x) = (x)(18 - 2x)(18 - 2x)$$

$$V(x) = x(18 - 2x)^2$$

$$\therefore V'(x) = (18 - 2x)^2 - 4x(18 - 2x)$$

$$= (18 - 2x)[18 - 2x - 4x]$$

$$= (18 - 2x)(18 - 6x)$$

$$= 6 \times 2(9 - x)(3 - x)$$

$$= 12(9 - x)(3 - x)$$
 [1 Mark]

Step 4:

$$\text{And, } V''(x) = 12[-(9 - x) - (3 - x)]$$

$$= -12(9 - x + 3 - x)$$

$$= -12(12 - 2x)$$

$$= -24(6 - x)$$
 [$\frac{1}{2}$ Mark]

Step 5:

$$\text{Now, } V'(x) = 0$$

$$12(9 - x)(3 - x) = 0$$

$$\Rightarrow x = 9 \text{ or } x = 3$$
 [$\frac{1}{2}$ Mark]

Step 6:

If $x = 9$, then, the length and the breadth will be 0.

$$\therefore x \neq 9.$$

$$\Rightarrow x = 3.$$

$$\text{Now, } V''(3) = -24(6 - 3) = -72 < 0$$

Hence by second derivative test, $x = 3$ is the point of maxima of V .

Therefore, if we remove a square of side 3 cm from each corner of the square tin and make a box from the remaining sheet, then the volume of the box obtained is the largest possible.

[1 Mark]