CBSE NCERT Solutions for Class 12 Maths Chapter 04

Back of Chapter Questions

Exercise 4. 1

1. Evaluate the determinant $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$

Solution:

Given determinant is $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$

Expanding along R_1, we get

$= \left(2 \times (-1)\right) - \left(4 \times (-5)\right)$

$= -2 + 20 = 18$

Hence, $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix} = 18$

2. Evaluate the determinants

(i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

(ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

Solution:

i) Given determinant is $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

Now, $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

$= (\cos\theta \times \cos\theta) - (\sin\theta \times (-\sin\theta))$

$= \cos^2\theta + \sin^2\theta = 1$

Hence $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = 1$

ii) Given determinant is $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

Now, $\begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix}$

$= (x^2-x+1)\times(x+1)-(x-1)\times(x+1)$

$= x^3+x^2-x^2-x+x+1-(x^2-1)$

$= x^3-x^2+2$

Hence, $\begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix} = x^3-x^2+2$

3. If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then show that $|2A| = 4|A|$

Solution:

Given that $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$

Now, $2A = \begin{bmatrix} 2\times1 & 2\times2 \\ 2\times4 & 2\times2 \end{bmatrix}$

$= \begin{bmatrix} 2 & 4 \\ 8 & 4 \end{bmatrix}$

$|2A| = \begin{vmatrix} 2 & 4 \\ 8 & 4 \end{vmatrix}$

Expanding along R_1, we get

$= 2\times4-4\times8 = 8-32 = -24$ ...(i)

And $4|A| = 4\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix}$

Expanding along R_1, we get

$= 4(1\times2-2\times4) = 4(-6) = -24$ ...(ii)

From the equation (i) and (ii),

we get $|2A| = 4|A|$

Hence proved.

4. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, then show that $|3A| = 27|A|$

Solution:

Given that $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$

$\Rightarrow 3A = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{bmatrix}$

Now, $|3A| = \begin{vmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{vmatrix}$

Expanding along R_1, we get

$= 3(36 - 0) - 0(0 - 0) + 3(0 - 0) = 108$ ...(i)

And $27|A| = 27 \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix}$

Expanding along R_1, we get

$= 27\{1(4 - 0) - 0(0 - 0) + 1(0 - 0)\} = 27(4) = 108$ ...(ii)

From the equation (i) and (ii),

we get $|3A| = 27|A|$

Hence proved.

5. Evaluate the determinants:

(i) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

(ii) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

(iii) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$

(iv) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

Solution:

(i) Given determinant is $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

Now, expanding the determinant along R_1, we get

$$\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix} = 3(0-5)+1(0+3)-2(0-0) = -15+3-0 = -12$$

Hence $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix} = -12$

Given determinant is $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

Now, expanding the determinant along R_1, we get

$$\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 3(1+6)+4(1+4)+5(3-2) = 21+20+5 = 46$$

Hence $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 46$

(iii) Given determinant is $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$

Now, expanding the determinant along R_1, we get

$$\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix} = 0(0+9)-1(0-6)+2(-3-0) = 0+6-6$$

So, the given determinant $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix} = 0$

Given determinant is $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

Now, expanding the determinant along R_1, we get

$$\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix} = 2(0-5)+1(0+3)-2(0-6) = -10+3+12 = 5$$

Hence, $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix} = 5$

6. If $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$, find $|A|$.

Solution:

Given that $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{vmatrix}$

Expanding the determinant along R_1, we get

$= 1(-9 + 12) - 1(-18 + 15) - 2(8 - 5) = 3 + 3 - 6 = 0$

Hence, $|A| = 0$

7. Find values of x, if

(i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

(ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

Solution:

(i) Given that $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

$\Rightarrow 2 - 20 = 2x^2 - 24$

$\Rightarrow x^2 = 3$

$\Rightarrow x = \pm\sqrt{3}$

Hence, the values of x are $\pm\sqrt{3}$

(ii) Given that $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

$\Rightarrow 10 - 12 = 5x - 6x$

$\Rightarrow -2 = -x$

$\Rightarrow x = 2$

Hence, the value of x is 2

8. If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then x is equal to:

(A) 6

(B) ± 6

(C) -6

(D) 0

Solution:

(B)

Given that $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$

$\Rightarrow x^2 - 36 = 36 - 36$

$\Rightarrow x^2 = 36$

$\Rightarrow x = \pm 6$

Hence, the option (B) is correct.

Exercise 4.2

Using the property of determinants and without expanding, prove that:

1. $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$

Solution:

$\text{LHS} = \begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix}$

$= \begin{vmatrix} x+a & a & x+a \\ y+b & b & y+b \\ z+c & c & z+c \end{vmatrix}$ [Applying $C_1 \to C_1 + C_2$]

$= 0 = \text{RHS} \quad [\because C_1 = C_3]$

Hence, LHS = RHS

2. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

Solution:

$$\text{LHS} = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

$$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} \text{[Applying } C_1 \to C_1 + C_2 + C_3]$$

$= 0 = \text{RHS}$ [$\because$ In column C_1 every element is zero.]

Hence, LHS = RHS

3. $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0$

Solution:

$$\text{LHS} = \begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 7 & 63 \\ 3 & 8 & 72 \\ 5 & 9 & 81 \end{vmatrix} \text{[Applying } C_3 \to C_3 - C_1]$$

$$= 9\begin{bmatrix} 2 & 7 & 7 \\ 3 & 8 & 8 \\ 5 & 9 & 9 \end{bmatrix} \quad \text{[Taking common 9 from } C_3]$$

$= 0 = \text{RHS}$ [$\because C_2 = C_3$]

Hence, LHS = RHS

4. $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$

Solution:

$$LHS = \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & bc & ab+bc+ca \\ 1 & ca & ab+bc+ca \\ 1 & ab & ab+bc+ca \end{vmatrix} \text{[Applying } C_3 \to C_3 + C_2]$$

$$= (ab+bc+ca) \begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix} \quad \text{[Taking } ab+bc+ca \text{ as common from } C_3]$$

$$= 0 = RHS \quad [\because C_1 = C_3]$$

Hence, LHS = RHS

5. $\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$

Solution:

$$LHS = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

$$= \begin{vmatrix} 2c & 2r & 2z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} \quad \text{[Applying } R_1 \to R_1 + R_2 - R_3]$$

$$= 2\begin{vmatrix} c & r & z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} \quad \text{[Taking 2 as common from } R_1]$$

$$= 2\begin{vmatrix} c & r & z \\ a & p & x \\ a+b & p+q & x+y \end{vmatrix} \quad \text{[Applying } R_2 \to R_2 - R_1]$$

$$= 2\begin{vmatrix} c & r & z \\ a & p & x \\ b & q & y \end{vmatrix} \quad \text{[Applying } R_3 \to R_3 - R_2]$$

$$= -2\begin{vmatrix} a & p & x \\ c & r & z \\ b & q & y \end{vmatrix} \quad \text{[Interchanging } R_1 \leftrightarrow R_2]$$

$$= 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = RHS \quad \text{[Interchanging } R_2 \leftrightarrow R_3]$$

Hence, LHS = RHS

6. $\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$

Solution:

$LHS = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$

$= \begin{vmatrix} 0 & a & -b \\ -ab & 0 & -bc \\ ab & ac & 0 \end{vmatrix}$ [Applying $R_2 \to bR_2$ and $R_3 \to aR_3$]

$= \begin{vmatrix} 0 & a & -b \\ 0 & ac & -bc \\ ab & ac & 0 \end{vmatrix}$ [Applying $R_2 \to R_2 + R_3$]

$= ab(-abc + abc)$ [Expanding along C_1]

$= ab(0) = 0 = RHS$

Hence, LHS = RHS

7. $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$

Solution:

$LHS = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$

$= abc \begin{vmatrix} -a & a & a \\ b & -b & b \\ c & c & -c \end{vmatrix}$ [Taking a, b, c as common from C_1, C_2, C_3 respectively]

$= a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$ [Taking a, b, c as common from R_1, R_2, R_3 respectively]

$= a^2b^2c^2 \begin{vmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$ [Applying $C_1 \to C_1 + C_2$]

$= a^2b^2c^2\{2(1+1)\}$ [Expanding along C_1]

$= 4a^2b^2c^2 = RHS$

Hence, LHS = RHS

8. Using properties of determinants, show that

(i) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

Solution:

(i) $LHS = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

$= \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$ [Applying $R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$]

$= (a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$ [Taking common a − b from R_1 and b − c from R_2]]

$= (a-b)(b-c)\{1(b+c-a-b)\}$ [Expanding along C_1]

$= (a-b)(b-c)(c-a) = RHS$

Hence, LHS = RHS

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

$LHS = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix}$ [By $C_1 \to C_1 - C_2, C_2 \to C_2 - C_3$]

$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \end{vmatrix}$ [Taking common a − b from C_1 and b − c from C_2]

$= (a-b)(b-c)\{1(b^2+bc+c^2)-(a^2+ab+b^2)\}$ [Expanding along R_1]

$= (a-b)(b-c)\{c^2-a^2+bc-ab\}$

$= (a-b)(b-c)\{(c-a)(c+a)+b(c-a)\}$

$= (a-b)(b-c)(c-a)\{c+a+b\}$

$= (a-b)(b-c)(c-a)(a+b+c) = RHS$

Hence, LHS = RHS

9. Using properties of determinants, show that

$$\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$$

Solution:

$$LHS = \begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & x^3 & xyz \\ y^2 & y^3 & xyz \\ z^2 & z^3 & xyz \end{vmatrix} \text{[Applying } R_1 \to xR_1, R_2 \to yR_2, R_3 \to zR_3\text{]}$$

$$= xyz \begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix} \quad \text{[Taking xyz as common from } C_3\text{]}$$

$$= xyz \begin{vmatrix} x^2-y^2 & x^3-y^3 & 0 \\ y^2-z^2 & y^3-z^3 & 0 \\ z^2 & z^3 & 1 \end{vmatrix} \quad \text{[Applying } R_1 \to R_1 - R_2, R_2 \to R_2 - R_3\text{]}$$

$$= xyz(x-y)(y-z) \begin{vmatrix} x+y & x^2+xy+y^2 & 0 \\ y+z & y^2+yz+z^2 & 0 \\ z^2 & z^3 & 1 \end{vmatrix} \quad \text{[Taking } x-y \text{ as common from } R_1 \text{ and } y-z \text{ from } R_2\text{]}$$

$$= xyz(x-y)(y-z)\{(x+y)(y^2+yz+z^2)-(y+z)(x^2+xy+y^2)\} \quad \text{[Expanding along } C_3\text{]}$$

$$= xyz(x-y)(y-z)\{xy^2+xyz+xz^2+y^3+y^2z+yz^2-(x^2y+xy^2+y^3+x^2z+xyz+y^2z)\}$$

$$= xyz(x-y)(y-z)\{xz^2+yz^2-x^2y-x^2z\}$$

$$= xyz(x-y)(y-z)\{xz^2-x^2z+yz^2-x^2y\}$$

$$= xyz(x-y)(y-z)\{xz(z-x)+y(z^2-x^2)\}$$

$$= xyz(x-y)(y-z)(z-x)\{xz+y(z+x)\}$$

$$= (x-y)(y-z)(z-x)(xy+yz+zx) = RHS$$

Hence, LHS = RHS

10. Using properties of determinants, show that

(i) $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$

(ii) $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k)$

Solution:

(i) $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$

$LHS = \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$

$= \begin{vmatrix} 5x+4 & 2x & 2x \\ 5x+4 & x+4 & 2x \\ 5x+4 & 2x & x+4 \end{vmatrix}$ [Applying $C_1 \to C_1 + C_2 + C_3$]

$= (5x+4)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x+4 & 2x \\ 1 & 2x & x+4 \end{vmatrix}$ [Taking $5x+4$ as common from C_1]

$= (5x+4)\begin{vmatrix} 0 & x-4 & 0 \\ 0 & 4-x & x-4 \\ 1 & 2x & x+4 \end{vmatrix}$ [Applying $R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$]

$= (5x+4)\{(x-4)(x-4) - (4-x)0\}$ [Expanding along C_1]

$= (5x+4)(4-x)^2 = RHS$

Hence, LHS = RHS

(iii) $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k)$

$LHS = \begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$

$= \begin{vmatrix} 3y+k & y & y \\ 3y+k & y+k & y \\ 3y+k & y & y+k \end{vmatrix}$ [Applying $C_1 \to C_1 + C_2 + C_3$]

$= (3y+k)\begin{vmatrix} 1 & y & y \\ 1 & y+k & y \\ 1 & y & y+k \end{vmatrix}$ [Taking $3y+k$ as common from C_1]

$= (3y+k)\begin{vmatrix} 0 & -k & 0 \\ 0 & k & -k \\ 1 & y & y+k \end{vmatrix}$ [Applying $R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$]

$= (3y + k)\{(-k)(-k) - (k)0\}$ [Expanding along C_1]

$= (3y + k)k^2 = RHS$

Hence, LHS = RHS

11. Using properties of determinants, show that

(i) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

(ii) $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$

Solution:

(i) $LHS = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$ [Applying $R_1 \to R_1 + R_2 + R_3$]

$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$ [Taking a + b + c as commong from R_1]

$= (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ a+b+c & -a-b-c & 2b \\ 0 & a+b+c & c-a-b \end{vmatrix}$ [By $C_1 \to C_1 - C_2, C_2 \to C_2 - C_3$]

$= (a+b+c)\{(a+b+c)^2 - 0\}$ [Expanding along R_1]

$= (a+b+c)^3 = RHS$

$(ii) \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$

$LHS = \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$

$= \begin{vmatrix} 2(x+y+z) & x & y \\ 2(x+y+z) & y+z+2x & y \\ 2(x+y+z) & x & z+x+2y \end{vmatrix}$ [Applying $C_1 \to C_1 + C_2 + C_3$]

$$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix} \quad [\text{Taking } 2(x+y+z) \text{ common from } C_1]$$

$$= 2(x+y+z)\begin{vmatrix} 0 & -(x+y+z) & 0 \\ 0 & x+y+z & -(x+y+z) \\ 1 & x & z+x+2y \end{vmatrix} \quad [\text{By } R_1 \to R_1 - R_2, R_2 \to R_2 - R_3]$$

$= 2(x+y+z)\{(x+y+z)^2 - 0\}$ [Expanding along C_1]

$= 2(x+y+z)^3 = RHS$

Hence, LHS = RHS

12. Using properties of determinants, show that

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$$

Solution:

$$LHS = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1+x+x^2 & x & x^2 \\ 1+x+x^2 & 1 & x \\ 1+x+x^2 & x^2 & 1 \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 + C_2 + C_3]$$

$$= (1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & x \\ 1 & x^2 & 1 \end{vmatrix} \quad [\text{Taking } 1+x+x^2 \text{ as common from } C_1]$$

$$= (1+x+x^2)\begin{vmatrix} 0 & x-1 & x^2-x \\ 0 & 1-x^2 & x-1 \\ 1 & x^2 & 1 \end{vmatrix} \quad [\text{Applying } R_1 \to R_1 - R_2, R_2 \to R_2 - R_3]$$

$$= (1+x+x^2)(1-x)^2\begin{vmatrix} 0 & -1 & -x \\ 0 & 1-x & -1 \\ 1 & x^2 & 1 \end{vmatrix} \quad [\text{Taking } 1-x \text{ as common from } R_1 \text{ and } R_2]$$

$= (1+x+x^2)(1-x)^2\{1+x(1+x)\}$ [Expanding along C_1]

$= (1+x+x^2)(1-x^2)(1+x+x^2) = (1-x^3)^2 = RHS$

Hence, LHS = RHS

13. Using properties of determinants, show that

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$$

Solution:

$$LHS = \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

$$= \frac{1}{a}\begin{vmatrix} 1+a^2-b^2 & 2ab & -2ab \\ 2ab & 1-a^2+b^2 & 2a^2 \\ 2b & -2a & a-a^3-ab^2 \end{vmatrix} \quad [\text{Applying } C_3 \to aC_3]$$

$$= \frac{1}{a}\begin{vmatrix} 1+a^2-b^2 & 2ab & 0 \\ 2ab & 1-a^2+b^2 & 1+a^2+b^2 \\ 2b & -2a & -a-a^3-ab^2 \end{vmatrix} \quad [\text{Applying } C_3 \to C_3 + C_2]$$

$$= \frac{1+a^2+b^2}{a}\begin{vmatrix} 1+a^2-b^2 & 2ab & 0 \\ 2ab & 1-a^2+b^2 & 1 \\ 2b & -2a & -a \end{vmatrix} \quad [\text{Taking } 1+a^2+b^2 \text{ as common from } C_3]$$

$$= \frac{1+a^2+b^2}{a^2}\begin{vmatrix} 1+a^2-b^2 & 2ab & 0 \\ 2a^2b & a-a^3+ab^2 & a \\ 2b & -2a & -a \end{vmatrix} \quad [\text{Applying } R_2 \to aR_2]$$

$$= \frac{1+a^2+b^2}{a^2}\begin{vmatrix} 1+a^2-b^2 & 2ab & 0 \\ 2a^2b+2b & -a-a^3+ab^2 & 0 \\ 2b & -2a & -a \end{vmatrix} \quad [\text{Applying } R_2 \to R_2 + R_3]$$

$$= (1+a^2+b^2)\begin{vmatrix} 1+a^2-b^2 & 2b & 0 \\ 2a^2b+2b & -1-a^2+b^2 & 0 \\ 2b & -2 & -1 \end{vmatrix} \quad [\text{Taking } a \text{ as common from } C_2 \text{ and } C_3]$$

$$= (1+a^2+b^2)(-1)\{(1+a^2-b^2)(-1-a^2+b^2) - 2b(2a^2b+2b)\} \quad [\text{Expanding along } C_3]$$

$$= -(1+a^2+b^2)\{-1-a^2+b^2-a^2-a^4+a^2b^2+b^2+a^2b^2-b^4-4a^2b^2-4b^2\}$$

$$= (1+a^2+b^2)\{1+a^4+4+2a^2+2a^2b^2+2b^2\}$$

$$= (1+a^2+b^2)(1+a^2+b^2)^2 = (1+a^2+b^2)^3 = RHS$$

Hence, LHS = RHS

14. Using properties of determinants, show that

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2$$

Solution:

$$LHS = \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} a^2+a & a^2b & a^2c \\ ab^2 & b^3+b & b^2c \\ c^2a & c^2b & c^3+c \end{vmatrix} \quad [\text{Applying } R_1 \to aR_1, R_3 \to bR_3, R_3 \to cR_3]$$

$$= \frac{abc}{abc}\begin{vmatrix} a^2+1 & a^2 & a^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix} \quad [\text{Taking a as common from } C_1, b \text{ from } C_2 \text{ and c from } C_3]$$

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix} \quad [\text{By } R_1 \to R_1 + R_2 + R_3]$$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix} \quad [\text{Taking } 1+a^2+b^2+c^2 \text{ as common from } R_1]$$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & b^2 \\ 0 & -1 & c^2+1 \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 - C_2, C_2 \to C_2 - C_3]$$

$$= (1+a^2+b^2+c^2)\{1-0\} \quad [\text{Expanding along } R_1]$$

$$= 1+a^2+b^2+c^2 = RHS$$

Hence, LHS = RHS

15. Let A be a square matrix of order 3×3, then $|kA|$ is equal to:

(A) $k|A|$

(B) $k^2|A|$

(C) $k^3|A|$

(D) $3k|A|$

Solution:

If B be a square matrix of order $n \times n$, then $|kB| = k^{n-1}|B|$

Therefore, $|kA| = k^{3-1}|A| = k^2|A|$

Hence, the option (B) is correct.

16. Which of the following is correct

(A) Determinant is a square matrix

(B) Determinant is a number associated to a matrix

(C) Determinant is a number associated to a square matrix

(D) None of these

Solution:

Determinant is a number associated to a square matrix

Hence, the option (C) is correct.

Exercise 4.3

1. Find area of the triangle with vertices at the point given in each of the following:

(i) $(1,0), (6,0), (4,3)$

(ii) $(2,7), (1,1), (10,8)$

(iii) $(-2,-3), (3,2), (-1,-8)$

Solution:

i) Given vertices of the triangle are $(1,0), (6,0), (4,3)$

We know, area of triangle $= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Area of triangle $= \frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 6 & 0 & 1 \\ 4 & 3 & 1 \end{vmatrix}$

$= \frac{1}{2}[1(0-3) - 0(6-4) + 1(18-0)] = \frac{1}{2}(15) = 7.5$ square units

Given vertices of the triangle are $(2,7), (1,1), (10,8)$

We know, area of triangle $=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Area of triangle $=\frac{1}{2}\begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix}$

$=\frac{1}{2}[2(1-8)-7(1-10)+1(8-10)]=\frac{1}{2}(47)=25.5$ square units

Given vertices of the triangle are $(-2,-3),(3,2),(-1,-8)$

We know, area of triangle $=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Area of triangle $=\frac{1}{2}\begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix}$

$=\frac{1}{2}[-2(2+8)+3(3+1)+1(-24+2)]=\frac{1}{2}(-30)=15$

Area of triangle $=15$ square units

2. Show that points $A(a,b+c),B(b,c+a),C(c,a+b)$ are collinear.

Solution:

If the points $A(a,b+c),B(b,c+a)$ and $C(c,a+b)$ are collinear, then the area of triangle ABC will be zero.

We know, area of triangle $=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Area of triangle $ABC=\frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$

$=\frac{1}{2}\begin{vmatrix} a & a+b+c & 1 \\ b & a+b+c & 1 \\ c & a+b+c & 1 \end{vmatrix}$ [Applying $C_2 \to C_1+C_2$]

$=\frac{1}{2}(a+b+c)\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix}$ [Taking $a+b+c$ as common from C_2]

$=0 \quad [\because C_1=C_3]$

Hence, the points $A(a,b+c),B(b,c+a)$ and $C(c,a+b)$ are collinear.

3. Find values of k if area of triangle is 4 square units and vertices are

(i) $(k, 0), (4, 0), (0, 2)$

(ii) $(-2, 0), (0, 4), (0, k)$

Solution:

(i)Given vertices of the triangle are $(k, 0)$, $(4, 0)$, $(0, 2)$

We know, area of triangle $= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Area of triangle $= \frac{1}{2}\begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix}$

$= \frac{1}{2}[k(0-2) - 0(4-0) + 1(8-0)] = \frac{1}{2}(-2k+8) = -k+4$

According to question, area of triangle $ABC = 4$ square units

Therefore, $|-k+4| = 4 \quad \Rightarrow -k+4 = \pm 4$

$\Rightarrow -k+4 = 4 \quad \text{or} \quad -k+4 = -4$

$\Rightarrow k = 0 \quad \text{or} \quad k = 8$

Hence, the value of k are 0 and 8.

ii) Given vertices of the triangle are $(-2, 0)$, $(0, 4)$, $(0, k)$

We know, area of triangle $= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Now, area of triangle $= \frac{1}{2}\begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & k & 1 \end{vmatrix}$

$= \frac{1}{2}[-2(4-k) - 0(0-0) + 1(0-0)] = \frac{1}{2}(-8+2k) = -4+k$

According to question, Area of triangle $ABC = 4$ square units

Therefore, $|-4+k| = 4 \quad \Rightarrow -4+k = \pm 4$

$\Rightarrow -4+k = 4 \quad \text{or} \quad -4+k = -4$

$\Rightarrow k = 8 \quad \text{or} \quad k = 0$

Hence, the value of k are 0 and 8.

4. (i) Find equation of line joining $(1, 2)$ and $(3, 6)$ using determinants.

(ii) Find equation of line joining $(3, 1)$ and $(9, 3)$ using determinants.

Solution:

(i) Let, $P(x, y)$ be any point lie on the line joining $A(1, 2)$ and $B(3, 6)$. Hence, the points A, B and P will be collinear and area of triangle ABC will be zero.

Therefore, area of triangle $ABP = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ x & y & 1 \end{vmatrix} = 0$

$\Rightarrow \frac{1}{2}[1(6 - y) - 2(3 - x) + 1(3y - 6x)] = 0$

$\Rightarrow 6 - y - 6 + 2x + 3y - 6x = 0$

$\Rightarrow -4x + 2y = 0$

$\Rightarrow 2x = y$

Hence the equation of line joining the points $(1, 2)$ and $(3, 6)$ is $2x - y = 0$

(ii) Let, $P(x, y)$ be any point lie on the line joining $A(3, 1)$ and $B(9, 3)$. Hence, the points A, B and P will be collinear and area of triangle ABC will be zero.

Therefore, area of triangle $ABP = \frac{1}{2}\begin{vmatrix} 3 & 1 & 1 \\ 9 & 3 & 1 \\ x & y & 1 \end{vmatrix} = 0$

$\Rightarrow \frac{1}{2}[3(3 - y) - 1(9 - x) + 1(9y - 3x)] = 0$

$\Rightarrow 9 - 3y - 9 + x + 9y - 3x = 0$

$\Rightarrow -2x + 6y = 0$

$\Rightarrow x = 3y$

Hence the equation of line joining the points $(3, 1)$ and $(9, 3)$ is $x - 3y = 0$

5. If area of triangle is 35 sq. units with vertices $(2, -6)$, $(5, 4)$ and $(k, 4)$. Then k is

(A) 12

(B) -2

(C) $-12, -2$

(D) $12, -2$

Solution:

Given vertices of the triangle are $(2, -6)$, $(5, 4)$, $(k, 4)$

We know, area of triangle $= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Hence, area of triangle $= \frac{1}{2}\begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ k & 4 & 1 \end{vmatrix}$

$= \frac{1}{2}[2(4-4) + 6(5-k) + 1(20-4k)] = \frac{1}{2}(30 - 6k + 20 - 4k) = 25 - 5k$

According to question, area of triangle $= 35$ square units

Therefore, $|25 - 5k| = 35$

$\Rightarrow 25 - 5k = \pm 35$

$\Rightarrow 25 - 5k = 35 \quad \text{or} \quad 25 - 5k = -35$

$\Rightarrow k = \dfrac{-10}{5} = -2 \quad \text{or} \quad k = \dfrac{60}{5} = 12$

Hence, the option (D) is correct.

Exercise 4. 4

1. Write Minors and Cofactors of the elements of following determinants:

(i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$

(ii) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

Solution:

(i)

Given determinant is $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$

We know, the minor of element a_{ij} is M_{ij} and the cofactor is $A_{ij} = (-1)^{i+j} M_{ij}$, therefore,

The minor of element a_{11} is $M_{11} = 3$ and the cofactor is $A_{11} = (-1)^{1+1} M_{11} = 3$

The minor of element a_{12} is $M_{12} = 0$ and the cofactor is $A_{12} = (-1)^{1+2} M_{12} = 0$

The minor of element a_{21} is $M_{21} = -4$ and the cofactor is $A_{21} = (-1)^{2+1} M_{21} = 4$

The minor of element a_{22} is $M_{22} = 2$ and the cofactor is $A_{22} = (-1)^{2+2} M_{22} = 2$

(ii)

Given determinant is $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

We know, the minor of element a_{ij} is M_{ij} and the cofactor is $A_{ij} = (-1)^{i+j}M_{ij}$, therefore,

The minor of element a_{11} is $M_{11} = d$ and the cofactor is $A_{11} = (-1)^{1+1}M_{11} = d$

The minor of element a_{12} is $M_{12} = b$ and the cofactor is $A_{12} = (-1)^{1+2}M_{12} = -b$

The minor of element a_{21} is $M_{21} = c$ and the cofactor is $A_{21} = (-1)^{2+1}M_{21} = -c$

The minor of element a_{22} is $M_{22} = a$ and the cofactor is $A_{22} = (-1)^{2+2}M_{22} = a$

2. Write Minors and Cofactors of the elements of following determinants:

(i) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

(ii) $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

Solution:

(i)

Given determinant is $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

We know, minor of an element a_{ij} of a determinant is the determinant obtained by deleting its i^{th} row and j^{th} column in which element a_{ij} lies. Minor of an element a_{ij} is denoted by M_{ij}.

Hence,

$M_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$

$M_{12} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 - 0 = 0$

$M_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0$

$M_{21} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 - 0 = 0$

$M_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$

$M_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0$

$M_{31} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0 - 0 = 0$

$M_{32} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0$

$M_{33} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$

And we know, cofactor of an element a_{ij}, is denoted by A_{ij} is defined by $A_{ij} = (-1)^{i+j} M_{ij}$, therefore

$A_{11} = (-1)^{1+1} M_{11} = 1 \quad A_{12} = (-1)^{1+2} M_{12} = 0 \quad A_{13} = (-1)^{1+3} M_{13} = 0$
$A_{21} = (-1)^{2+1} M_{21} = 0 \quad A_{22} = (-1)^{2+2} M_{22} = 1 \quad A_{23} = (-1)^{2+3} M_{233} = 0$
$A_{31} = (-1)^{3+1} M_{31} = 0 \quad A_{32} = (-1)^{3+2} M_{32} = 0 \quad A_{33} = (-1)^{3+3} M_{33} = 1$

(ii)

Given determinant is $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

We know, minor of an element a_{ij} of a determinant is the determinant obtained by deleting its i^{th} row and j^{th} column in which element a_{ij} lies. Minor of an element a_{ij} is denoted by M_{ij}.

Hence,

$M_{11} = \begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = 10 + 1 = 11$

$M_{12} = \begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = 6 - 0 = 6$

$M_{13} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3$

$M_{21} = \begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} = 0 - 4 = -4$

$M_{22} = \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2$

$M_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$

$M_{31} = \begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix} = 0 - 20 = -20$

$M_{32} = \begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix} = -1 - 12 = -13$

$M_{33} = \begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} = 5 - 0 = 5$

And we know, cofactor of an element a_{ij}, is denoted by A_{ij} is defined by $A_{ij} = (-1)^{i+j} M_{ij}$, therefore

$$A_{11} = (-1)^{1+1}M_{11} = 11 \quad A_{12} = (-1)^{1+2}M_{12} = -6 \quad A_{13} = (-1)^{1+3}M_{13} = 3$$
$$A_{21} = (-1)^{2+1}M_{21} = 4 \quad A_{22} = (-1)^{2+2}M_{22} = 2 \quad A_{23} = (-1)^{2+3}M_{233} = -1$$
$$A_{31} = (-1)^{3+1}M_{31} = -20 \quad A_{32} = (-1)^{3+2}M_{32} = 13 \quad A_{33} = (-1)^{3+3}M_{33} = 5$$

3. Using cofactors of elements of second row, evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

Solution:

We know, $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix} = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$

Here, $a_{21} = 2, a_{22} = 0, a_{23} = 1$ and

$A_{21} = (-1)^{2+1}\begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix} = -(9 - 16) = 7$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix} = 15 - 8 = 7$

$A_{23} = (-1)^{2+3}\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = -(10 - 3) = -7$

Therefore, $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 2(7) + 0(7) + 1(-7) = 7$

4. Using Cofactors of elements of third column, evaluate $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$.

Solution:

We know, $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix} = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$

Here, $a_{13} = yz, a_{23} = zx, a_{33} = xy$ and

$A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix} = z - y$

$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & x \\ 1 & z \end{vmatrix} = -(z - x) = x - z$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix} = y - x$

Therefore, $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix} = yz(z - y) + zx(x - z) + xy(y - x)$

$= yz^2 - y^2z + zx^2 - xz^2 + xy^2 - x^2y$

$= zx^2 - x^2y - xz^2 + xy^2 + yz^2 - y^2z$

$= x^2(z - y) - x(z^2 - y^2) + yz(z - y)$

$= (z - y)[x^2 - x(z + y) + yz]$

$= (z - y)[x^2 - xz - xy + yz]$

$= (z - y)[x(x - z) - y(x - z)]$

$= (x - z)(z - y)(x - y)$

$= (x - y)(y - z)(z - x)$

Hence, $\begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix} = (x - y)(y - z)(z - x)$

5. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ A_{ij} is Cofactors of a_{ij}, then value of Δ is given by

(A) $a_{11}A_{11} + a_{12}A_{32} + a_{13}A_{33}$

(B) $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$

(C) $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$

(D) $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

Solution:

The value of $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ is given by: $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

Hence, the option (D) is correct.

Exercise 4.5

1. Find adjoint of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Solution:

Given matrix is $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

We know, the adjoint of a square matrix $A = [a_{ij}]_{n\times n}$ is defined as the transpose of the matrix $[A_{ij}]_{n\times n}$, where A_{ij} is the cofactor of the element a_{ij}. Adjoint of the matrix A is denoted by $adj\ A$.

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

therefore, $A_{11} = 4,\ A_{12} = -3,\quad A_{21} = -2,\quad A_{22} = 1$

Hence, adjoint of matrix $A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

2. Find adjoint of the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

Solution:

Given matrix is $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

We know, the adjoint of a square matrix $A = [a_{ij}]_{n\times n}$ is defined as the transpose of the matrix $[A_{ij}]_{n\times n}$, where A_{ij} is the cofactor of the element a_{ij}. Adjoint of the matrix A is denoted by $adj\ A$.

Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$, therefore

$A_{11} = 3 \qquad A_{12} = -12 \quad A_{13} = 6$
$A_{21} = 1 \qquad A_{22} = 5 \quad A_{23} = 2$
$A_{31} = -11 \quad A_{32} = -1 \quad A_{33} = 5$

Hence, adjoint of matrix $A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$

3. Verify $A(adj\ A) = (adj\ A).A = |A|.I$ for the matrix $\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$

Solution:

Given matrix is $\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$

Let $A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$,

therefore, $A_{11} = -6 \quad A_{12} = 4 \quad A_{21} = -3 \quad A_{22} = 2$

$|A| = -12 + 12 = 0$

$adj\ A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$

$A(adj\ A) = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} -12+12 & -6+6 \\ 24-24 & 12-12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$(adj\ A).A = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix} = \begin{bmatrix} -12+12 & -18+18 \\ 8-8 & 12-12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$|A|.I = 0.\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Hence, $A(adj\ A) = (adj\ A).A = |A|.I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

4. Verify $A(adj\ A) = (adj\ A).A = |A|.I$ for the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

Solution:

Given matrix is $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

Here, $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$, therefore, $|A| = 1(0-0) + 1(9+2) + 2(0-0) = 11$

$A_{11} = 0 \quad A_{12} = -11 \quad A_{13} = 0$
$A_{21} = 3 \quad A_{22} = 1 \quad A_{23} = -1$
$A_{31} = 2 \quad A_{32} = 8 \quad A_{33} = 3$

Adjoint of matrix $A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$

$A(adj\ A) = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 0+11+0 & 3-1-2 & 2-8+6 \\ 0+0+0 & 9+0+2 & 6+0-6 \\ 0+0+0 & 3+0-3 & 2+0+9 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$

Now, $(adj\ A).A = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 0+9+2 & 0+0+0 & 0-6+6 \\ -11+3+8 & 11+0+0 & -22-2+24 \\ 0-3+3 & 0+0+0 & 0+2+9 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$|A|.I = 11.\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

Hence, $A(adj\ A) = (adj\ A).A = |A|.I = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$

5. Find the inverse of the matrix (if it exists)

$\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$

Solution:

Given matrix is $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$

Let $A = \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$

Therefore, $A_{11} = 3 \quad A_{12} = -4 \quad A_{21} = 2 \quad A_{22} = 2$

And $|A| = 6 + 8 = 14 \neq 0 \Rightarrow A^{-1}$ exists.

We know, $A^{-1} = \frac{1}{|A|} adj\ A =$

$\Rightarrow A^{-1} = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}$

=> $\frac{1}{14}\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$

Hence, the inverse of the matrix $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$ is $\frac{1}{14}\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$

6. Find the inverse of the matrix (if it exists)

$\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$

Solution:

Given matrix is $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$

Let $A = \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$

Therefore, $A_{11} = 2 \quad A_{12} = 3 \quad A_{21} = -5 \quad A_{22} = -1$

$|A| = -2 + 15 = 13 \neq 0 \Rightarrow A^{-1}$ exists.

We know $A^{-1} = \frac{1}{|A|} adj\ A$

$\Rightarrow A^{-1} = \frac{1}{|A|} \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$

Hence, the inverse of the matrix $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$ is $\frac{1}{13} \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$

7. Find the inverse of the matrix (if it exists)

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

Solution:

Given matrix is $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

Therefore, $|A| = 1(10 - 0) - 2(0 - 0) + 3(0 - 0) = 10 \neq 0 \Rightarrow A^{-1}$ exists.

$A_{11} = 10 \quad A_{12} = 0 \quad A_{13} = 0$
$A_{21} = -10 \quad A_{22} = 5 \quad A_{23} = 0$
$A_{31} = 2 \quad A_{32} = -4 \quad A_{33} = 2$

We know $A^{-1} = \frac{1}{|A|} adj\ A$

$\Rightarrow A^{-1} = \frac{1}{|A|} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

$= \frac{1}{10} \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$

Hence, the inverse of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ is $\frac{1}{10} \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$

8. Find the inverse of the matrix (if it exists)

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$$

Solution:

Given matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$,

Therefore, $|A| = 1(-3-0) - 0(-3-0) + 0(6-15) = -3 \neq 0 \Rightarrow A^{-1}$ exists.

$A_{11} = -3 \quad A_{12} = 3 \quad A_{13} = -9$
$A_{21} = 0 \quad A_{22} = -1 \quad A_{23} = -2$
$A_{31} = 0 \quad A_{32} = 0 \quad A_{33} = 3$

We know, $A^{-1} = \frac{1}{|A|} adj\ A$

$\Rightarrow A^{-1} = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$

Hence, the inverse of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$ is $\frac{1}{3}\begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$

9. Find the inverse of the matrix (if it exists)

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$$

Solution:

Given matrix is $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$

Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$

Therefore, $|A| = 2(-1-0) - 1(4-0) + 3(8-7) = -3 \neq 0 \Rightarrow A^{-1}$ exists.

$A_{11} = -1 \quad A_{12} = -4 \quad A_{13} = 1$
$A_{21} = 5 \quad A_{22} = 23 \quad A_{23} = -11$
$A_{31} = 3 \quad A_{32} = 12 \quad A_{33} = -6$

We know, $A^{-1} = \frac{1}{|A|} adj\ A$

$\Rightarrow A^{-1} = \frac{1}{|A|} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

$= \frac{1}{3} \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$

Hence, the inverse of the matrix $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$ is $\frac{1}{3} \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$

10. Find the inverse of the matrix (if it exists)

$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

Solution:

Given matrix is $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

Therefore, $[A] = 1(8-6) + 1(0+9) + 2(0-6) = -1 \neq 0 \Rightarrow A^{-1}$ exists.

$A_{11} = 2 \quad A_{12} = -9 \quad A_{13} = -6$
$A_{21} = 0 \quad A_{22} = -2 \quad A_{23} = -1$
$A_{31} = -1 \quad A_{32} = 3 \quad A_{33} = 2$

We know, $A^{-1} = \frac{1}{|A|} adj\ A$

$\Rightarrow A^{-1} = \frac{1}{|A|} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

$= \frac{1}{-1}\begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

Hence, the inverse of the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$ is $\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

11. Find the inverse of the matrix (if it exists)

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$

Solution:

Given matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$, therefore

Now, $|A| = 1(-\cos^2\alpha - \sin^2\alpha) + 0(0-0) + 0(0-0) = -1 \neq 0$

$\Rightarrow A^{-1}$ exists. Therefore

$A_{11} = 1 \quad A_{12} = 0 \quad A_{13} = 0$
$A_{21} = 0 \quad A_{22} = -\cos\alpha \quad A_{23} = -\sin\alpha$
$A_{31} = 0 \quad A_{32} = -\sin\alpha \quad A_{33} = \cos\alpha$

We know, $A^{-1} = \frac{1}{|A|} adj\, A$

$\Rightarrow A^{-1} = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

$= \frac{1}{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$

$\Rightarrow A^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$

Hence, the inverse of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$ is $\begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$

12. Let $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$. Verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Solution:

Given matrices are $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$

Here, $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$, therefore, $A_{11} = 5 \quad A_{12} = -2 \quad A_{21} = -7 \quad A_{22} = 3$

Now, $|A| = 15 - 14 = 1 \neq 0 \Rightarrow A^{-1}$ exists.

We know, $A^{-1} = \frac{1}{|A|} adj\ A$

$$\Rightarrow A^{-1} = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}$$

$$= \frac{1}{1}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$, therefore, $B_{11} = 9 \quad B_{12} = -7 \quad B_{21} = -8 \quad B_{22} = 6$

$|B| = 54 - 56 = -2 \neq 0 \Rightarrow B^{-1}$ exists.

$$B^{-1} = \frac{1}{|B|} adj\ B$$

$$= \frac{1}{|B|}\begin{bmatrix} B_{11} & B_{21} \\ B_{12} & B_{22} \end{bmatrix}$$

$$= \frac{1}{-2}\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{9}{2} & 4 \\ \frac{7}{2} & -3 \end{bmatrix}$$

Now, $B^{-1}A^{-1} = \begin{bmatrix} -9/2 & 4 \\ 7/2 & -3 \end{bmatrix}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} -\frac{45}{2} - 8 & \frac{63}{2} + 12 \\ \frac{35}{2} + 6 & -\frac{49}{2} - 9 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{61}{2} & \frac{87}{2} \\ \frac{47}{2} & -\frac{67}{2} \end{bmatrix} \text{...(i)}$$

And $AB = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 18 + 49 & 24 + 63 \\ 12 + 35 & 16 + 45 \end{bmatrix} = \begin{bmatrix} 67 & 87 \\ 47 & 61 \end{bmatrix}$

$|AB| = 67 \times 61 - 87 \times 47 = 4087 - 4089 = -2 \neq 0 \Rightarrow (AB)^{-1}$ exists.

$AB_{11} = 61 \quad AB_{12} = -47 \quad AB_{21} = -87 \quad AB_{22} = 67$

$(AB)^{-1} = \frac{1}{|AB|} adj\ AB = \frac{1}{|AB|}\begin{bmatrix} AB_{11} & AB_{21} \\ AB_{12} & AB_{22} \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix} = \begin{bmatrix} -\frac{61}{2} & \frac{87}{2} \\ \frac{47}{2} & -\frac{67}{2} \end{bmatrix}$ ...(ii)

From (i) and (ii)

$(AB)^{-1} = B^{-1}A^{-1}$

Hence verified.

13. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ show that $A^2 - 5A + 7I = 0$. Hence, find A^{-1}.

Solution:

Given matrix is $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$LHS = A^2 - 5A + 7I = AA - 5A + 7I$

$= \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O = RHS$

$\Rightarrow A^2 - 5A + 7I = O$

$\Rightarrow A^2 - 5A = -7I$

Multiplying by A^{-1}(because $|A| \neq 0$)

$\Rightarrow AAA^{-1} - 5AA^{-1} = -7IA^{-1}$

$\Rightarrow AI - 5I = -7A^{-1} \quad [\because AA^{-1} = I]$

$\Rightarrow 7A^{-1} = 5I - A = 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

$\Rightarrow A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

14. For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the numbers a and b such that $A^2 + aA + bI = O$.

Solution:

Given matrix is $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

And $A^2 + aA + bI = O$

$$\Rightarrow \begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}\begin{bmatrix}3 & 2\\1 & 1\end{bmatrix} + a\begin{bmatrix}3 & 2\\1 & 1\end{bmatrix} + b\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}9+2 & 6+2\\3+1 & 2+1\end{bmatrix} + \begin{bmatrix}3a & 2a\\a & a\end{bmatrix} + \begin{bmatrix}b & 0\\0 & b\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}11+3a+b & 8+2a+0\\4+a+0 & 3+a+b\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

$\Rightarrow 4 + a = 0$

$\Rightarrow a = -4$

and $3 + a + b = 0$

$\Rightarrow b = -3 - a = -3 + 4$

$= 1$

Hence, $a = -4,\ b = 1$

15. For the matrix $A = \begin{bmatrix}1 & 1 & 1\\1 & 2 & -3\\2 & -1 & 3\end{bmatrix}$

show that $A^3 - 6A^2 + 5A + 11I = 0$. Hence, find A^{-1}.

Solution:

Given matrix is $A = \begin{bmatrix}1 & 1 & 1\\1 & 2 & -3\\2 & -1 & 3\end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix}1 & 1 & 1\\1 & 2 & -3\\2 & -1 & 3\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\1 & 2 & -3\\2 & -1 & 3\end{bmatrix}$$

$$= \begin{bmatrix}1+1+2 & 1+2-1 & 1-3+3\\1+2-6 & 1+4+3 & 1-6-9\\2-1+6 & 2-2-3 & 2+3+9\end{bmatrix} = \begin{bmatrix}4 & 2 & 1\\-3 & 8 & -14\\7 & -3 & 14\end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix}4 & 2 & 1\\-3 & 8 & -14\\7 & -3 & 14\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\1 & 2 & -3\\2 & -1 & 3\end{bmatrix}$$

$$= \begin{bmatrix}4+2+2 & 4+4-1 & 4-6+3\\-3+8-28 & -3+16+14 & -3-24-42\\7-3+28 & 7-6-14 & 7+9+42\end{bmatrix} = \begin{bmatrix}8 & 7 & 1\\-23 & 27 & -69\\32 & -13 & 58\end{bmatrix}$$

$\text{LHS} = A^3 - 6A^2 + 5A + 11I$

$$= \begin{bmatrix}8 & 7 & 1\\-23 & 27 & -69\\32 & -13 & 58\end{bmatrix} - 6\begin{bmatrix}4 & 2 & 1\\-3 & 8 & -14\\7 & -3 & 14\end{bmatrix} + 5\begin{bmatrix}1 & 1 & 1\\1 & 2 & -3\\2 & -1 & 3\end{bmatrix} + 11\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 8-24+5+11 & 7-12+5+0 & 1-6+5+0 \\ -23+18+5+0 & 27-48+10+11 & -69+84-15+0 \\ 32-42+10+0 & -13+18-5+0 & 58-84+15+11 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O = RHS$$

Now, $A^3 - 6A^2 + 5A + 11\,I = O$

$\Rightarrow A^3 - 6A^2 + 5A = -11\,I$

Multiplying by A^{-1} (because $|A| \neq 0$)

$\Rightarrow A^2AA^{-1} - 6AAA^{-1} + 5AA^{-1} = -11A^{-1}$

$\Rightarrow A^2I - 6AI + 5I = -11A^{-1}$ $\quad [\because\ AA^{-1} = I]$

$\Rightarrow 11A^{-1} = -A^2 + 6A - 5I$

$$\Rightarrow 11A^{-1} = -\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 6\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow 11A^{-1} = \begin{bmatrix} -4 & -2 & -1 \\ 3 & -8 & 14 \\ -7 & 3 & -14 \end{bmatrix} + \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\Rightarrow 11A^{-1} = \begin{bmatrix} -4+6-5 & -2+6+0 & -1+6+0 \\ 3+6-0 & -8+12-5 & 14-18+0 \\ -7+12+0 & 3-6+0 & -14+18-5 \end{bmatrix} = \begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

Hence, $A^{-1} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$

16. If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

verify that $A^3 - 6A^2 + 9A - 4I = O$ and hence find A^{-1}.

Solution:

$$A^2 = A.A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+5+6 & -5-10-6 & 5+5+12 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$\text{LHS} = A^3 - 6A^2 + 9A - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 22-36+18-4 & -21+30-9+0 & 21-30+9+0 \\ -21+30-9-0 & 22-36+18-4 & -21+30-9+0 \\ 21-30+9-0 & -21+30-9-0 & 22-36+18-4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O = RHS$$

$\Rightarrow A^3 - 6A^2 + 9A - 4I = O \quad \Rightarrow A^3 - 6A^2 + 9A = 4I$

Post multiplying by A^{-1}(because $|A| \neq 0$)

$A^2AA^{-1} - 6AAA^{-1} + 9AA^{-1} = 4IA^{-1}$

$\Rightarrow A^2I - 6AI + 9I = 4A^{-1} \quad$ [Because $AA^{-1} = I$]

$\Rightarrow 4A^{-1} = A^2 - 6A + 9I$

$$\Rightarrow 4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow 4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\Rightarrow 4A^{-1} = \begin{bmatrix} 6-12+9 & -5+6+0 & 5-6+0 \\ -5+6+0 & 6-12+9 & -5+6+0 \\ 5-6+0 & -5+6+0 & 6-12+9 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Hence, $A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

17. Let A be a non-singular square matrix of order 3×3. Then $|adj\ A|$ is equal to:

(A) $|A|$

(B) $|A|^2$

(C) $|A|^3$

(D) $3|A|$

Solution:

Given that A be a non-singular square matrix of order 3×3.

We know that $adj\ A = |A|I$

$\Rightarrow (adj\ A)A = |A|\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow |(adj\ A)A| = |A|\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix} = |A|^3$

$\Rightarrow |adj\ A| = |A|^2$,

Hence, the option (B) is correct.

18. If A is an invertible matrix of order 2, then $\det(A^{-1})$ is equal to:

(A) $\det(A)$

(B) $\frac{1}{\det(A)}$

(C) 1

(D) 0

Solution:

Given that the matrix A is invertible, hence, $A^{-1} = \frac{1}{|A|} adj\ A$

The order of matrix is 2, so, let $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Therefore, $|A| = ad - bc$ and $adj\ A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} adj\ A = A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{|A|} & -\frac{b}{|A|} \\ -\frac{c}{|A|} & \frac{a}{|A|} \end{bmatrix}$

$\det(A^{-1}) = |A^{-1}| = \begin{vmatrix} \frac{d}{|A|} & -\frac{b}{|A|} \\ -\frac{c}{|A|} & \frac{a}{|A|} \end{vmatrix}$

$= \frac{1}{|A|^2}\begin{vmatrix} d & -b \\ -c & a \end{vmatrix} = \frac{1}{|A|^2}(ad - bc) = \frac{1}{|A|^2}|A| = \frac{1}{|A|}$

Hence, the option (B) is correct.

Exercise 4.6

1. Examine the consistency of the system of equations

$x + 2y = 2$

$2x + 3y = 3$

Solution:

The given system of equations are $x + 2y = 2$ and $2x + 3y = 3$

This system of equations can be written as $AX = B$.

Where, $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Now, $|A| = 3 - 4 = -1 \neq 0 \Rightarrow A$ is non-singular and so A^{-1} exists.

Hence, the system of equations are consistent.

2. Examine the consistency of the system of equations

$2x - y = 5$

$x + y = 4$

Solution:

The given system of equations are $2x - y = 5$ and $x + y = 4$

This system of equations can be written as $AX = B$.

Where, $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

Now, $|A| = 2 + 1 = 3 \neq 0 \Rightarrow A$ is non-singular and so A^{-1} exists.

Hence, the system of equations are consistent.

3. Examine the consistency of the system of equations

$x + 3y = 5$

$2x + 6y = 8$

Solution:

The given system of equations: $x + 3y = 5$ and $2x + 6y = 8$

This system of equations can be written as $AX = B$, where

$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

Now, $|A| = 6 - 6 = 0 \Rightarrow A$ is a singular matrix and so A^{-1} does not exist.

Now, $adj\ A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

$(adj\ A)\ B = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 30 - 24 \\ -10 + 8 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \neq O$

So, there is no solutions of the given system of equations.

Hence, the system of equations are inconsistent.

4. Examine the consistency of the system of equations

$x + y + z = 1$

$2x + 3y + 2z = 2$

$ax + ay + 2az = 4$

Solution:

The given system of equations are $x + y + z = 1$, $2x + 3y + 2z = 2$ and $ax + ay + 2az = 4$

This system of equations can be written as $AX = B$, where

$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ a & a & 2a \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Now, $|A| = 1(6a - 2a) - 1(4a - 2a) + 1(2a - 3a) = a \neq 0$

$\Rightarrow A$ is non-singular and so A^{-1} exists.

Hence, the system of equations are consistent.

5. Examine the consistency of the system of equations

$3x - y - 2z = 2$

$2y - z = -1$

$3x - 5y = 3$

Solution:

The given system of equations are $3x - y - 2z = 2$, $2y - z = -1$ and $3x - 5y = 3$

This system of equations can be written as $AX = B$, where

$A = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

$|A| = 3(0 - 5) + 1(0 + 3) - 2(0 - 6) = -15 + 3 + 12 = 0$

$\Rightarrow A$ is a singular matrix and so A^{-1} does not exists. Now,

$A_{11} = -5 \quad A_{12} = -3 \quad A_{13} = -6$
$A_{21} = 10 \quad A_{22} = 6 \quad A_{23} = 12$
$A_{31} = 5 \quad A_{32} = 3 \quad A_{33} = 6$

Now, $adj\ A = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}$

Also $(adj\ A)B = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -10 - 10 + 15 \\ -6 - 6 + 9 \\ -12 - 12 + 16 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ -6 \end{bmatrix} \neq O$

So, there is no solutions of the given system of equations.

Hence, the system of equations are inconsistent.

6. Examine the consistency of the system of equations

$5x - y + 4z = 5$

$2x + 3y + 5z = 2$

$5x - 2y + 6z = -1$

Solution:

The given system of equations are $5x - y + 4z = 5$, $2x + 3y + 5z = 2$ and $5x - 2y + 6z = -1$

This system of equations can be written as $AX = B$, where

$A = \begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$

Now, $|A| = 5(18 + 10) + 1(12 - 25) + 4(-4 - 15) = 140 - 13 - 76 = 51 \neq 0$

$\Rightarrow A$ is non-singular and so A^{-1} exists.

Hence, the system of equations are consistent.

7. Solve system of linear equations, using matrix method

$5x + 2y = 4$

$7x + 3y = 5$

Solution:

The given system of equations are $5x + 2y = 4$ and $7x + 3y = 5$

This system of equations can be written as $AX = B$, where

$A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$|A| = 15 - 14 = 1 \neq 0 \Rightarrow A$ is non-singular and so A^{-1} exists.

Hence, the system of equations are consistent.

Now, $A_{11} = 3 \quad A_{12} = -7 \quad A_{21} = -2 \quad A_{22} = 5$

We know, $A^{-1} = \frac{1}{|A|} adj\ A$

$\Rightarrow A^{-1} = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \frac{1}{1}\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$

Also $X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 - 10 \\ -28 + 25 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \quad \Rightarrow x = 2, \quad y = -3$

8. Solve system of linear equations, using matrix method

$2x - y = -2$

$3x + 4y = 3$

Solution:

The given system of equations are $2x - y = -2$ and $3x + 4y = 3$

This system of equations can be written as $AX = B$, where

$A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$|A| = 8 + 3 = 11 \neq 0 \Rightarrow A$ is non-singular and so A^{-1} exists. Now,

Hence, the system of equations are consistent.

Now, $A_{11} = 4 \quad A_{12} = -3 \quad A_{21} = 1 \quad A_{22} = 2$

$A^{-1} = \frac{1}{|A|} adj\, A$

$= \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$

$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -8+3 \\ 6+6 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{5}{11} \\ \frac{12}{11} \end{bmatrix} \quad \Rightarrow x = -\frac{5}{11}, \quad y = \frac{12}{11}$

9. Solve system of linear equations, using matrix method

$4x - 3y = 3$

$3x - 5y = 7$

Solution:

The given system of equations are $4x - 3y = 3$ and $3x - 5y = 7$

This system of equations can be written as $AX = B$, where

$A = \begin{bmatrix} 4 & -3 \\ 3 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

$|A| = -20 + 9 = -11 \neq 0 \Rightarrow A$ is non-singular and so A^{-1} exists.

Hence, the system of equations are consistent.

Now, $A_{11} = -5 \quad A_{12} = -3 \quad A_{21} = 3 \quad A_{22} = 4$

$A^{-1} = \frac{1}{|A|} adj\ A$

$= \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \frac{1}{-11}\begin{bmatrix} -5 & 3 \\ -3 & 4 \end{bmatrix}$

Now, $X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{11}\begin{bmatrix} -5 & 3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{11}\begin{bmatrix} -15+21 \\ -9+28 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{6}{11} \\ -\frac{19}{11} \end{bmatrix} \Rightarrow x = -\frac{6}{11}, y = -\frac{19}{11}$

10. Solve system of linear equations, using matrix method

$5x + 2y = 3$

$3x + 2y = 5$

Solution:

The given system of equations are $5x + 2y = 3$ and $3x + 2y = 5$

This system of equations can be written as $AX = B$, where

$A = \begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$|A| = 10 - 6 = 4 \neq 0 \Rightarrow A$ is non-singular and so A^{-1} exists.

Hence, the system of equations are consistent.

Now, $A_{11} = 2 \quad A_{12} = -3 \quad A_{21} = -2 \quad A_{22} = 5$

$A^{-1} = \frac{1}{|A|} adj\ A$

$= \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$

$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 6-10 \\ -9+25 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{4}{4} \\ \frac{16}{4} \end{bmatrix} \quad \Rightarrow x = -1, \quad y = 4$

11. Solve system of linear equations, using matrix method

$2x + y + z = 1$

$x - 2y - z = \frac{3}{2}$

$3y - 5z = 9$

Solution:

The given system of equations are $2x + y + z = 1$, $x - 2y - z = \frac{3}{2}$ and $3y - 5z = 9$

This system of equations can be written as $AX = B$, where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ \frac{3}{2} \\ 9 \end{bmatrix}$$

$|A| = 2(10 + 3) - 1(-5 - 0) + 1(3 - 0) = 26 + 5 + 3 = 34 \neq 0$

$\Rightarrow A$ is non-singular and so A^{-1} exists. Now,

$A_{11} = 13 \quad A_{12} = 5 \quad A_{13} = 3$
$A_{21} = 8 \quad A_{22} = -10 \quad A_{23} = -6$
$A_{31} = 1 \quad A_{32} = 3 \quad A_{33} = -5$

$$A^{-1} = \frac{1}{|A|} adj\ A$$

$$= \frac{1}{34}\begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}$$

Now, $X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ \frac{3}{2} \\ 9 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 13 + 12 + 9 \\ 5 - 15 + 27 \\ 3 - 9 - 45 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 34 \\ 17 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ -\frac{3}{2} \end{bmatrix} \Rightarrow x = 1, y = \frac{1}{2}, z = -\frac{3}{2}$$

12. Solve system of linear equations, using matrix method

$x - y + z = 4$

$2x + y - 3z = 0$

$x + y + z = 2$

Solution:

The given system of equations are $x - y + z = 4$, $2x + y - 3z = 0$ and

$x+y+z=2$

This system of equations can be written as $AX=B$, where

$$A=\begin{bmatrix}1 & -1 & 1\\ 2 & 1 & -3\\ 1 & 1 & 1\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \text{ and } B=\begin{bmatrix}4\\ 0\\ 2\end{bmatrix}$$

$|A|=1(1+3)+1(2+3)+1(2-1)=4+5+1=10\neq 0$

$\Rightarrow A$ is non-singular and so A^{-1} exists. Now,

$A_{11}=4 \quad A_{12}=-5 \quad A_{13}=1$
$A_{21}=2 \quad A_{22}=0 \quad A_{23}=-2$
$A_{31}=2 \quad A_{32}=5 \quad A_{33}=3$

Now, $A^{-1}=\frac{1}{|A|}adj\ A$

$$=\frac{1}{10}\begin{bmatrix}4 & 2 & 2\\ -5 & 0 & 5\\ 1 & -2 & 3\end{bmatrix}$$

Also $X=A^{-1}B\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{10}\begin{bmatrix}4 & 2 & 2\\ -5 & 0 & 5\\ 1 & -2 & 3\end{bmatrix}\begin{bmatrix}4\\ 0\\ 2\end{bmatrix}$

$$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{10}\begin{bmatrix}16+0+4\\ -20+0+10\\ 4+0+6\end{bmatrix}\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{10}\begin{bmatrix}20\\ -10\\ 10\end{bmatrix}=\begin{bmatrix}2\\ -1\\ 1\end{bmatrix}$$

$\Rightarrow x=2, y=-1, z=1$

13. Solve system of linear equations, using matrix method

$2x+3y+3z=5$

$x-2y+z=-4$

$3x-y-2z=3$

Solution:

The given system of equations are $2x+3y+3z=5,\ x-2y+z=-4$ and $3x-y-2z=3$

This system of equations can be written as $AX=B$, where

$$A=\begin{bmatrix}2 & 3 & 3\\ 1 & -2 & 1\\ 3 & -1 & -2\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \text{ and } B=\begin{bmatrix}5\\ -4\\ 3\end{bmatrix}$$

$|A|=2(4+1)-3(-2-3)+3(-1+6)=10+15+15=40\neq 0$

$\Rightarrow A$ is non-singular and so A^{-1} exists. Now,

$A_{11} = 5 \quad A_{12} = 5 \quad A_{13} = 5$
$A_{21} = 3 \quad A_{22} = -13 \quad A_{23} = 11$
$A_{31} = 9 \quad A_{32} = 1 \quad A_{33} = -7$

Now, $A^{-1} = \frac{1}{|A|} adj\ A$

$$= \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

Also, $X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 25 - 12 + 27 \\ 25 + 52 + 3 \\ 25 - 44 - 21 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$\Rightarrow x = 1, y = 2, z = -1$

14. Solve system of linear equations, using matrix method

$x - y + 2z = 7$

$3x + 4y - 5z = -5$

$2x - y + 3z = 12$

Solution:

The given system of equations are $x - y + 2z = 7$, $3x + 4y - 5z = -5$ and $2x - y + 3z = 12$

This system of equations can be written as $AX = B$, where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$|A| = 1(12 - 5) + 1(9 + 10) + 2(-3 - 8) = 7 + 19 - 22 = 4 \neq 0$

$\Rightarrow A$ is non-singular and so A^{-1} exists. Now,

$A_{11} = 7 \quad A_{12} = -19 \quad A_{13} = -11$
$A_{21} = 1 \quad A_{22} = -1 \quad A_{23} = -1$
$A_{31} = -3 \quad A_{32} = 11 \quad A_{33} = 7$

Now, $A^{-1} = \frac{1}{|A|} adj\ A$

$$= \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

Also, $X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}\begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 49 - 5 - 36 \\ -133 + 5 + 132 \\ -77 + 5 + 84 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 4 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$\Rightarrow x = 1, y = 1, z = 3$

15. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}. Using A^{-1} solve the system of equations

$2x - 3y + 5z = 11$

$3x + 2y - 4z = -5$

$x + y - 2z = -3$

Solution:

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$$

$|A| = 2(-4 + 4) + 3(-6 + 4) + 5(3 - 2) = 0 - 6 + 5 = -1 \neq 0$

$\Rightarrow A$ is non-singular and so A^{-1} exists. Now,

$$\begin{matrix} A_{11} = 0 & A_{12} = 2 & A_{13} = 1 \\ A_{21} = -1 & A_{22} = -9 & A_{23} = -5 \\ A_{31} = 2 & A_{32} = 23 & A_{33} = 13 \end{matrix}$$

We know, $A^{-1} = \frac{1}{|A|} adj\ A$

$$= \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$

The given system of equations:

$2x - 3y + 5z = 11$

$3x + 2y - 4z = -5$

$x + y - 2z = -3$

This system of equations can be written as $AX = B$, where

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0-5+6 \\ -22-45+69 \\ -11-25+39 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow x = 1, y = 2, z = 3$

16. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ₹60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is ₹90. The cost of 6 kg onion 2 kg wheat and 3 kg rice is ₹70. Find cost of each item per kg by matrix method.

Solution:

Let the cost of 1 kg of onion = ₹x,

Let the cost of 1 kg of wheat = ₹y and

Let the cost of 1 kg rice = ₹z

The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ₹60. So $4x + 3y + 2z = 60$

The cost of 2 kg onion, 4 kg wheat and 6 kg rice is ₹90. So $2x + 4y + 6z = 90$ and

The cost of 6 kg onion, 2 kg wheat and 3 kg rice is ₹70. So $6x + 2y + 3z = 70$

Therefore the system of equations are:

$4x + 3y + 2z = 60$

$2x + 4y + 6z = 90$

$6x + 2y + 3z = 70$

This system of equations can be written as $AX = B$, where

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$|A| = 4(12 - 12) - 3(6 - 36) + 2(4 - 24) = 0 + 90 - 40 = 50 \neq 0$

$\Rightarrow A$ is non-singular and so A^{-1} exists. Now,

$A_{11} = 0 \quad A_{12} = 30 \quad A_{13} = -20$
$A_{21} = -5 \quad A_{22} = 0 \quad A_{23} = 10$
$A_{31} = 10 \quad A_{32} = -20 \quad A_{33} = 10$

Now, $A^{-1}=\frac{1}{|A|}adj\,A=\frac{1}{50}\begin{bmatrix}0 & -5 & 10\\ 30 & 0 & -20\\ -20 & 10 & 10\end{bmatrix}$

Also $X=A^{-1}B \Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{50}\begin{bmatrix}0 & -5 & 10\\ 30 & 0 & -20\\ -20 & 10 & 10\end{bmatrix}\begin{bmatrix}60\\ 90\\ 70\end{bmatrix}$

$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{50}\begin{bmatrix}0-450+700\\ 1800+0-1400\\ -1200+900+700\end{bmatrix}$

$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{50}\begin{bmatrix}250\\ 400\\ 400\end{bmatrix}=\begin{bmatrix}5\\ 8\\ 8\end{bmatrix}$

$\Rightarrow x=5, y=8, z=8$

Hence,

the cost of 1 kg of onion is ₹7, the cost of 1 kg of wheat is ₹8 and the cost of 1 kg rice is ₹8.

Miscellaneous Exercise on 4

1. Prove that the determinant $\begin{vmatrix}x & \sin\theta & \cos\theta\\ -\sin\theta & -x & 1\\ \cos\theta & 1 & x\end{vmatrix}$ is independent of θ.

Solution:

Let the given determinant be $\Delta=\begin{vmatrix}x & \sin\theta & \cos\theta\\ -\sin\theta & -x & 1\\ \cos\theta & 1 & x\end{vmatrix}$

$=x(-x^2-1)-\sin\theta(-x\sin\theta-\cos\theta)+\cos\theta(-\sin\theta+x\cos\theta)$

$=-x^3-x+x\sin^2\theta+\sin\theta\cos\theta-\cos\theta\sin\theta+x\cos^2\theta$

$=-x^3-x+x(\sin^2\theta+\cos^2\theta)$

$=-x^3-x+x=-x^3$, which is independent of θ.

2. Without expanding the determinant, prove that $\begin{vmatrix}a & a^2 & bc\\ b & b^2 & ca\\ c & c^2 & ab\end{vmatrix}=\begin{vmatrix}1 & a^2 & a^3\\ 1 & b^2 & b^3\\ 1 & c^2 & c^3\end{vmatrix}$

Solution:

LHS $=\begin{vmatrix}a & a^2 & bc\\ b & b^2 & ca\\ c & c^2 & ab\end{vmatrix}$

$= \frac{1}{abc}\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix}$ [Applying $R_1 \to aR_1, R_2 \to bR_2, R_3 \to cR_3$]

$= \frac{abc}{abc}\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix}$ [Taking abc as common from C_3]

$= (-1)^1\begin{vmatrix} 1 & a^3 & a^2 \\ 1 & b^3 & b^2 \\ 1 & c^3 & c^2 \end{vmatrix}$ [Interchanging $C_1 \leftrightarrow C_3$]

$= (-1)^2\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$ [Interchanging $C_2 \leftrightarrow C_3$]

$= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \text{RHS}$

Hence proved.

3. Evaluate $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

Solution:

Given determinant is $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

$= -\sin\alpha(-\sin\alpha\sin^2\beta - \sin\alpha\cos^2\beta)$
$\qquad - 0(\cos\alpha\cos\beta\sin\alpha\sin\beta - \cos\alpha\sin\beta\sin\alpha\cos\beta)$

$+ \cos\alpha(\cos\alpha\cos^2\beta + \cos\alpha\sin^2\beta)$

$= \sin^2\alpha(\sin^2\beta + \cos^2\beta) + \cos^2\alpha(\cos^2\beta + \sin^2\beta)$ [Expanding along C_3]

$= \sin^2\alpha(\sin^2\beta + \cos^2\beta) + \cos^2\alpha(\cos^2\beta + \sin^2\beta)$

$= \sin^2\alpha + \sin^2\alpha = 1$ [$\because \sin^2\theta + \cos^2\theta = 1$]

4. If a, b and c are real numbers and

$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$

show that either $a + b + c = 0$ or $a = b = c$.

Solution:

Given that $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$ [Applying $R_1 \to R_1 + R_2 + R_3$]

$\Rightarrow 2(a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$ [Taking $2(a+b+c)$ as common from R_1][

$\Rightarrow 2(a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ c+a & b-c & b-a \\ a+b & c-a & c-b \end{vmatrix} = 0$ [Applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$]

$\Rightarrow 2(a+b+c)1[(b-c)(c-b)-(b-a)(c-a)] = 0$ [Expanding along R_1]

$\Rightarrow 2(a+b+c)[bc-b^2-c^2+bc-(bc-ba-ac+a^2)] = 0$

$\Rightarrow 2(a+b+c)[bc-b^2-c^2+ab+ca-a^2] = 0$

$\Rightarrow -(a+b+c)[2a^2+2b^2+2c^2-2ab-2bc-2ca] = 0$

$\Rightarrow -(a+b+c)[(a^2+b^2-2ab)+(b^2+c^2-2bc)+(c^2+a^2-2ca)] = 0$

$\Rightarrow -(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2] = 0$

$\Rightarrow a+b+c = 0$ or $(a-b)^2 = 0, (b-c)^2 = 0, (c-a)^2 = 0$

$\Rightarrow a+b+c = 0$ or $a-b = 0, \quad b-c = 0, \quad c-a = 0$

$\Rightarrow a+b+c = 0$ or a = b. $\quad b = c, \quad c = a$

Therefore, $a+b+c = 0$ or $a = b = c$

5. Solve the equation $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0, a \neq 0$

Solution:

Given that $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} 3x+a & 3x+a & 3x+a \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$ [Applying $R_1 \to R_1 + R_2 + R_3$]

$\Rightarrow (3x+a)\begin{vmatrix}1 & 1 & 1\\ x & x+a & x\\ x & x & x+a\end{vmatrix} = 0$ [Taking $(3x+a)$ as common from R_1]

$\Rightarrow (3x+a)\begin{vmatrix}1 & 0 & 0\\ x & a & 0\\ x & 0 & a\end{vmatrix} = 0$ [Applying $C_2 \to C_2 - C_1, C_3 \to C_3 + C_1$]

$\Rightarrow (3x+a)1[a^2 - 0] = 0$ [Expanding along R_1]

$\Rightarrow a^2(3x+a) = 0$

$\Rightarrow (3x+a) = 0$ [$\because a \neq 0$]

$\Rightarrow x = -\frac{a}{3}$

Value of x is $-\frac{a}{3}$

6. Prove that $\begin{vmatrix}a^2 & bc & ac+c^2\\ a^2+ab & b^2 & ac\\ ab & b^2+bc & c^2\end{vmatrix} = 4a^2b^2c^2$

Solution:

LHS $= \begin{vmatrix}a^2 & bc & ac+c^2\\ a^2+ab & b^2 & ac\\ ab & b^2+bc & c^2\end{vmatrix}$

$= \begin{vmatrix}2a^2 & 0 & 2ac\\ a^2+ab & b^2 & ac\\ ab & b^2+bc & c^2\end{vmatrix}$ [Applying $R_1 \to R_1 + R_2 - R_3$]

$= abc\begin{vmatrix}2a & 0 & 2a\\ a+b & b & a\\ b & b+c & c\end{vmatrix}$ [Taking a, b, c as common from C_1, C_2, C_3]

$= abc\begin{vmatrix}2a & 0 & 0\\ a+b & b & -b\\ b & b+c & c-b\end{vmatrix}$ [Applying $C_3 \to C_3 - C_1$]

$= (abc)2a[b(c-b) - (-b)(b+c)]$ [Expanding along R_1]

$= 2a^2bc[bc - b^2 + b^2 + bc] = 2a^2bc[2bc]$

$= 4a^2b^2c^2 =$ RHS

Hence proved

7. If $A^{-1} = \begin{bmatrix}3 & -1 & 1\\ -15 & 6 & -5\\ 5 & -2 & 2\end{bmatrix}$ and $B = \begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}$, find $(AB)^{-1}$.

Solution:

Given that $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

Here, $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

Therefore, $|B| = 1(3-0) - 2(-1-0) - 2(2-0) = 1 \neq 0 \Rightarrow B^{-1}$ exists.

$B_{11} = 3 \quad B_{12} = 1 \quad B_{13} = 2$
$B_{21} = 2 \quad B_{22} = 1 \quad B_{23} = 2$
$B_{31} = 6 \quad B_{32} = 2 \quad B_{33} = 5$

We know, $B^{-1} = \frac{1}{|B|} adj\ B$

$= \frac{1}{1}\begin{bmatrix} B_{11} & B_{21} & B_{31} \\ B_{12} & B_{22} & B_{32} \\ B_{13} & B_{23} & B_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

Also $(AB)^{-1} = B^{-1}A^{-1}$, therefore

$(AB)^{-1} = B^{-1}A^{\overset{-1}{z}}$

$= \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

$= \begin{bmatrix} 9-30+30 & -3+12-12 & 3-10+12 \\ 3-15+10 & -1+6-4 & 1-5+4 \\ 6-30+25 & -2+12-10 & 2-10+10 \end{bmatrix} = \begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Hence, $(AB)^{-1} = \begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

8. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$. Verify that

(i) $(adj\ A)^{-1} = adj(A^{-1})$

(ii) $(A^{-1})^{-1} = A$

Solution:

(i)

Given matrix is $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$, therefore

$|A| = 1(15-1) + 2(-10-1) + 1(-2-3) = -13 \neq 0 \Rightarrow A^{-1}$ exists.

$A_{11} = 14 \quad A_{12} = 11 \quad A_{13} = -5$
$A_{21} = 11 \quad A_{22} = 4 \quad A_{23} = -3$
$A_{31} = -5 \quad A_{32} = -3 \quad A_{33} = -1$

$$adj\ A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adj\ A = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \frac{1}{-13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix} \quad ...(i)$$

Let, $B = adj\ A$, so, $B = \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$, therefore

$|B| = 14(-4-9) - 11(-11-15) - 5(-33+20) = -182 + 286 + 65 = 169 \neq 0$

$\Rightarrow B^{-1}$ exists.

$B_{11} = -13 \quad B_{12} = 26 \quad B_{13} = -13$
$B_{21} = 26 \quad B_{22} = -39 \quad B_{23} = -13$
$B_{31} = -13 \quad B_{32} = -13 \quad B_{33} = -65$

$$B^{-1} = \frac{1}{|B|}\begin{bmatrix} B_{11} & B_{21} & B_{31} \\ B_{12} & B_{22} & B_{32} \\ B_{13} & B_{23} & B_{33} \end{bmatrix} = \frac{1}{169}\begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix}$$

$$\Rightarrow (adj\ A)^{-1} = \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix} \quad ...(ii)$$

Let, $C = A^{-1}$, so, $C = \frac{1}{-13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{14}{13} & -\frac{11}{13} & \frac{5}{13} \\ -\frac{11}{13} & -\frac{4}{13} & \frac{3}{13} \\ \frac{5}{13} & \frac{3}{13} & \frac{1}{13} \end{bmatrix}$, therefore

$C_{11} = -\frac{1}{13} \quad C_{12} = \frac{2}{13} \quad C_{13} = -\frac{1}{13}$
$C_{21} = \frac{2}{13} \quad C_{22} = -\frac{3}{13} \quad C_{23} = -\frac{1}{13}$
$C_{31} = -\frac{1}{13} \quad C_{32} = -\frac{1}{13} \quad C_{33} = -\frac{5}{13}$

$$Adj\ C = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} -\frac{1}{13} & \frac{2}{13} & -\frac{1}{13} \\ \frac{2}{13} & -\frac{3}{13} & -\frac{1}{13} \\ -\frac{1}{13} & -\frac{1}{13} & -\frac{5}{13} \end{bmatrix} = \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix}$$

$$\Rightarrow Adj\ C = adj\ (A^{-1}) = \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix} \quad ...(iii)$$

From the equations (ii) and (iii) we have, $(adj\ A)^{-1} = adj(A^{-1})$

(ii) From the equation (i), we have,

$$A^{-1} = \frac{1}{-13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

Let, $D = A^{-1}$, so, $D = \frac{1}{-13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{14}{13} & -\frac{11}{13} & \frac{5}{13} \\ -\frac{11}{13} & -\frac{4}{13} & \frac{3}{13} \\ \frac{5}{13} & \frac{3}{13} & \frac{1}{13} \end{bmatrix}$, therefore

$$|D| = -\left(\frac{1}{13}\right)^3 [14(-4-9) - 11(-11-15) - 5(-33+20)]$$

$= -\left(\frac{1}{13}\right)^3 (169) = -\frac{1}{13} \neq 0 \quad \Rightarrow D^{-1}$ exists.

$$D_{11} = -\frac{1}{13} \quad D_{12} = \frac{2}{13} \quad D_{13} = -\frac{1}{13}$$
$$D_{21} = \frac{2}{13} \quad D_{22} = -\frac{3}{13} \quad D_{23} = -\frac{1}{13}$$
$$D_{31} = -\frac{1}{13} \quad D_{32} = -\frac{1}{13} \quad D_{33} = -\frac{5}{13}$$

$$D^{-1} = \frac{1}{|D|}\begin{bmatrix} D_{11} & D_{21} & D_{31} \\ D_{12} & D_{22} & D_{32} \\ D_{13} & D_{23} & D_{33} \end{bmatrix} = \frac{1}{-1/13}\begin{bmatrix} -\frac{1}{13} & \frac{2}{13} & -\frac{1}{13} \\ \frac{2}{13} & -\frac{3}{13} & -\frac{1}{13} \\ -\frac{1}{13} & -\frac{1}{13} & -\frac{5}{13} \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

$$\Rightarrow D^{-1} = (A^{-1})^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = A$$

9. Evaluate $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$.

Solution:

Given determinant is $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

$$= \begin{vmatrix} 2(x+y) & y & x+y \\ 2(x+y) & x+y & x \\ 2(x+y) & x & y \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 + C_2 + C_3]$$

$$= 2(x+y)\begin{vmatrix} 1 & y & x+y \\ 1 & x+y & x \\ 1 & x & y \end{vmatrix} \quad [\text{Taking } 2(x+y) \text{ as common from } C_1]$$

$= 2(x+y)\begin{vmatrix} 0 & -x & y \\ 0 & y & x-y \\ 1 & y & y+k \end{vmatrix}$ [Applying $R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$]

$= 2(x+y)\{(-x)(x-y) - y.y\}$ [Expanding along C_1]

$= 2(x+y)(-x^2 + xy - y^2)$

$= -2(x+y)(x^2 - xy + y^2) = -2(x^3 + y^3)$

Hence, $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix} = -2(x^3 + y^3)$

10. Evaluate $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

Solution:

Given determinant is $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

$= \begin{vmatrix} 0 & -y & 0 \\ 0 & y & -x \\ 1 & x & x+y \end{vmatrix}$ [Applying $R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$]

$= \{(-y)(-x) - y.0\}$ [Expanding along C_1]

$= xy$

Hence, $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix} = xy$

11. Using properties of determinants, prove that:

$\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)$

Solution:

LHS $= \begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix}$

$= \begin{vmatrix} \alpha & \alpha^2 & \alpha+\beta+\gamma \\ \beta & \beta^2 & \alpha+\beta+\gamma \\ \gamma & \gamma^2 & \alpha+\beta+\gamma \end{vmatrix}$ [Applying $C_3 \to C_1 + C_3$]

$= (\alpha+\beta+\gamma) \begin{vmatrix} \alpha & \alpha^2 & 1 \\ \beta & \beta^2 & 1 \\ \gamma & \gamma^2 & 1 \end{vmatrix}$ [Taking $\alpha+\beta+\gamma$ as common from C_3]

$= (\alpha+\beta+\gamma) \begin{vmatrix} \alpha-\beta & \alpha^2-\beta^2 & 0 \\ \beta-\gamma & \beta^2-\gamma^2 & 0 \\ \gamma & \gamma^2 & 1 \end{vmatrix}$ [Applying $R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$]

$= (\alpha+\beta+\gamma)(\alpha-\beta)(\beta-\gamma) \begin{vmatrix} 1 & \alpha+\beta & 0 \\ 1 & \beta+\gamma & 0 \\ \gamma & \gamma^2 & 1 \end{vmatrix}$

[Taking $\alpha-\beta$ as common from R_1 and $\beta-\gamma$ as common from R_2]

$= (\alpha+\beta+\gamma)(\alpha-\beta)(\beta-\gamma)\{(\beta+\gamma)-(\alpha+\beta)\}$ [Expanding along C_3]

$= (\alpha+\beta+\gamma)(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha) = \text{RHS}$

Hence proved.

12. Using properties of determinants, prove that:

$\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x)$, where p is any scalar.

Solution:

$\text{LHS} = \begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$

$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix}$

$= (-1)^1 \begin{vmatrix} 1 & x^2 & x \\ 1 & y^2 & y \\ 1 & z^2 & z \end{vmatrix} + p \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix}$ [Taking p as common from C_3]

$= (-1)^2 \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + pxyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$ [Taking x, y, z as common from R_1, R_2, R_3 respectively]

$= (1 + pxyz)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$ $\left[\text{Taking } \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \text{ as common}\right]$

$= (1 + pxyz)\begin{vmatrix} 0 & x - y & x^2 - y^2 \\ 0 & y - z & y^2 - z^2 \\ 1 & z & z^2 \end{vmatrix}$ [Applying $R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$]

$= (1 + pxyz)(x - y)(y - z)\begin{vmatrix} 0 & 1 & x + y \\ 0 & 1 & y + z \\ 1 & z & z^2 \end{vmatrix}$ [Taking $x - y$ as common from R_1 and $y - z$ as common form R_2]

$= (1 + pxyz)(x - y)(y - z)\{(y + z) - (x + y)\}$ [Expanding along C_1]

$= (1 + pxyz)(x - y)(y - z)(z - x) = \text{RHS}$

Hence proved.

13. Using properties of determinants, prove that:

$$\begin{vmatrix} 3a & -a + b & -a + c \\ -b + a & 3b & -b + c \\ -c + a & -c + b & 3c \end{vmatrix} = 3(a + b + c)(ab + bc + ca)$$

Solution:

$\text{LHS} = \begin{vmatrix} 3a & -a + b & -a + c \\ -b + a & 3b & -b + c \\ -c + a & -c + b & 3c \end{vmatrix}$

$= \begin{vmatrix} a + b + c & -a + b & -a + c \\ a + b + c & 3b & -b + c \\ a + b + c & -c + b & 3c \end{vmatrix}$ [Applying $C_1 \to C_1 + C_2 + C_3$]

$= (a + b + c)\begin{vmatrix} 1 & -a + b & -a + c \\ 1 & 3b & -b + c \\ 1 & -c + b & 3c \end{vmatrix}$ [Taking $(a + b + c)$ as common from C_1]

$= (a + b + c)\begin{vmatrix} 0 & -a - 2b & -a + b \\ 0 & 2b + c & -b - 2c \\ 1 & -c + b & 3c \end{vmatrix}$ [Applying $R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$]

$= (a + b + c)\{(-a - 2b)(-b - 2c) - (2b + c)(-a + b)\}$ [Expanding along C_1]

$= (a + b + c)\left(ab + 2ac + 2b^2 + 4bc - (-2ab + 2b^2 - ac + bc)\right)$

$= (a + b + c)(3ab + 3bc + 3ca) = 3(a + b + c)(ab + bc + ca) = \text{RHS}$

Hence proved.

14. Using properties of determinants, prove that:

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$$

Solution:

$$\text{LHS} = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 3 & 7+3p \end{vmatrix} \quad [\text{Applying } R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1]$$

$$= 1\{1.(7+3p) - (3)(2+p)\} \quad [\text{Expanding along } C_1]$$

$$= 7 + 3p - 6 - 3p = 1 = \text{RHS}$$

Hence proved.

15. Using properties of determinants, prove that:

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix} = 0$$

Solution:

$$\text{LHS} = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$$

$$= \begin{vmatrix} \sin\alpha & \cos\alpha\cos\delta - \sin\alpha\sin\delta & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta\cos\delta - \sin\beta\sin\delta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma\cos\delta - \sin\gamma\sin\delta & \cos(\gamma+\delta) \end{vmatrix} [\text{Applying } C_2 \to \cos\delta C_2 - \sin\delta\, C_1]$$

$$= \begin{vmatrix} \sin\alpha & \cos(\alpha+\delta) & \cos(\alpha+\delta) \\ \sin\beta & \cos(\beta+\delta) & \cos(\beta+\delta) \\ \sin\gamma & \cos(\gamma+\delta) & \cos(\gamma+\delta) \end{vmatrix}$$

$$= 0 = \text{RHS} \quad [\because C_2 = C_3]$$

Hence proved.

16. Solve the system of equations:

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4$$

$$\frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1$$

$$\frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2$$

Solution:

The given system of equations are

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4$$

$$\frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1$$

$$\frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2$$

This system of equations can be written as $AX = B$, where

$$A=\begin{bmatrix}2 & 3 & 10\\ 4 & -6 & 5\\ 6 & 9 & -20\end{bmatrix}, X=\begin{bmatrix}1/x\\ 1/y\\ 1/z\end{bmatrix} \text{ and } B=\begin{bmatrix}4\\ 1\\ 2\end{bmatrix}$$

Now, $|A| = 2(120-45)-3(-80-30)+10(36+36) = 150+330+720 = 1200 \neq 0$
$\Rightarrow A^{-1}$ exists.

Therefore

$$\begin{array}{lll} A_{11}=75 & A_{12}=110 & A_{13}=72 \\ A_{21}=150 & A_{22}=-100 & A_{23}=0 \\ A_{31}=75 & A_{32}=30 & A_{33}=-24 \end{array}$$

We know, $A^{-1}=\frac{1}{|A|}adj\ A=\frac{1}{1200}\begin{bmatrix}75 & 150 & 75\\ 110 & -100 & 30\\ 72 & 0 & -24\end{bmatrix}$

Also, $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix}1/x\\ 1/y\\ 1/z\end{bmatrix}=\frac{1}{1200}\begin{bmatrix}75 & 150 & 75\\ 110 & -100 & 30\\ 72 & 0 & -24\end{bmatrix}\begin{bmatrix}4\\ 1\\ 2\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\frac{1}{x}\\ \frac{1}{y}\\ \frac{1}{z}\end{bmatrix}=\frac{1}{1200}\begin{bmatrix}300+150+150\\ 440-100+60\\ 288+0-48\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{5} \end{bmatrix}$$

$$\Rightarrow \frac{1}{x} = \frac{1}{2}, \quad \frac{1}{y} = \frac{1}{3}, \quad \frac{1}{z} = \frac{1}{5} \quad \Rightarrow x = 2, y = 3, z = 5$$

17. Choose the correct answer.

If a, b, c are in A.P., then the determinant $\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$ is:

(A) 0

(B) 1

(C) x

(D) $2x$

Solution:

Given that a, b, c are in A.P and determinant is $\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$

$$= \begin{vmatrix} x+2 & x+3 & x+2a \\ 0 & 0 & 2(2b-a-c) \\ x+4 & x+5 & x+2c \end{vmatrix} \quad [\text{Applying } R_2 \to 2R_2 - (R_1 - R_3)]$$

$$= \begin{vmatrix} x+2 & x+3 & x+2a \\ 0 & 0 & 0 \\ x+4 & x+5 & x+2c \end{vmatrix} \quad [\because a, b, c \text{ are in AP, therefore } 2b = a + c]$$

$$= 0 \quad [\because \text{All the elements of } R_2 \text{ is zero}]$$

Hence, the option (A) is correct.

18. Choose the correct answer.

If x, y, z are nonzero real numbers, then the inverse of matrix $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is:

(A) $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

(B) $xyz\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

(C) $\frac{1}{xyz}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$

(D) $\frac{1}{xyz}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution:

Given matrix is $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$

Now, $|A| = x(yz - 0) - 0(0 - 0) + 0(0 - 0) = xyz \neq 0 \Rightarrow A^{-1}$ exists. Therefore,

$A_{11} = yz \quad A_{12} = 0 \quad A_{13} = 0$
$A_{21} = 0 \quad A_{22} = xz \quad A_{23} = 0$
$A_{31} = 0 \quad A_{32} = 0 \quad A_{33} = xy$

We know, $A^{-1} = \frac{1}{|A|} adj\ A$

$= \frac{1}{xyz}\begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{bmatrix} = \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

Hence, the option (A) is correct.

19. Choose the correct answer.

Let $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, where $0 \leq \theta \leq 2\pi$ then:

(A) $\det(A) = 0$

(B) $\det(A) \in (2, \infty)$

(C) $\det(A) \in (2, 4)$

(D) $\det(A) \in [2, 4]$

Solution:

Given matrix is $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$

$= 1(1 + \sin^2\theta) + \sin\theta(-\sin\theta + \sin\theta) + 1(\sin^2\theta + 1)$ [Expanding along C_1]

$= 2(1 + \sin^2\theta)$

Now, given that: $0 \leq \theta \leq 2\pi$

$\Rightarrow 0 \leq \sin\theta \leq 1$

$\Rightarrow 0 \leq \sin^2\theta \leq 1$

$\Rightarrow 1 \leq 1 + \sin^2\theta \leq 2$

$\Rightarrow 2 \leq 2(1 + \sin^2\theta) \leq 4$

$\Rightarrow \det(A) \in [2, 4]$

Hence, the option (D) is correct.