CBSE NCERT Solutions for Class 12 Maths Chapter 02

EXERCISE 2.1

1. Find the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$

Solution:

Let $sin^{-1}\left(-\frac{1}{2}\right) = y$, then $\sin y = -\frac{1}{2} = -\sin\left(\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{6}\right)$

We know that the range of the principal value of

$\sin^{-1}x$ is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

Hence, the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$ is $-\frac{\pi}{6}$.

2. Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Solution:

Let $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = y$, then $\cos y = \frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right)$

We know that the range of the principal value of

$\cos^{-1}x$ is $[0,\pi]$ and $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

Hence, the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is $\frac{\pi}{6}$.

3. Find the principal value of $\operatorname{cosec}^{-1}(2)$

Solution:

Let $\operatorname{cosec}^{-1}(2) = y$. then, $\operatorname{cosec} y = 2 = \operatorname{cosec}\left(\frac{\pi}{6}\right)$

We know that the range of the principal value of

$\operatorname{cosec}^{-1}x$ is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$ and $\operatorname{cosec}\left(\frac{\pi}{6}\right)=2.$

Hence, the principal value of $\operatorname{cosec}^{-1}(2)$ is $\frac{\pi}{6}$.

4. Find the principal value of $\tan^{-1}(-\sqrt{3})$

Solution:

Let $\tan^{-1}(-\sqrt{3})=y$, then $\tan y=-\sqrt{3}=-\tan\frac{\pi}{3}=\tan\left(-\frac{\pi}{3}\right)$

We know that the range of the principal value of

$\tan^{-1}x$ is $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and $\tan\left(-\frac{\pi}{3}\right)=-\sqrt{3}$

Hence, the principal value of $\tan^{-1}(-\sqrt{3})$ is $-\frac{\pi}{3}$.

5. Find the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$

Solution:

Let $\cos^{-1}\left(-\frac{1}{2}\right)=y$, then,

$\cos y=-\frac{1}{2}=-\cos\frac{\pi}{3}=\cos\left(\pi-\frac{\pi}{3}\right)=\cos\left(\frac{2\pi}{3}\right)$

We know that the range of the principal value of

$\cos^{-1}x$ is $[0,\pi]$ and $\cos\left(\frac{2\pi}{3}\right)=-\frac{1}{2}$

Hence, the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is $\frac{2\pi}{3}$.

6. Find the principal value of $\tan^{-1}(-1)$

Solution:

Let $\tan^{-1}(-1)=y$. Then, $\tan y=-1=-\tan\left(\frac{\pi}{4}\right)=\tan\left(-\frac{\pi}{4}\right)$

We know that the range of the principal value of

$\tan^{-1}x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan\left(-\frac{\pi}{4}\right) = -1$

Hence, the principal value of $\tan^{-1}(-1)$ is $-\frac{\pi}{4}$.

7. Find the principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

Solution:

Let $sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = y$, then $\sec y = \frac{2}{\sqrt{3}} = sec\left(\frac{\pi}{6}\right)$

We know that the range of the principal value of x in

$\sec^{-1}x$ is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ and $\sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}$.

Hence, the principal value of $sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ is $\frac{\pi}{6}$.

8. Find the principal value of $\cot^{-1}(\sqrt{3})$

Solution:

Let $\cot^{-1}\sqrt{3} = y$, then $\cot y = \sqrt{3} = \cot\left(\frac{\pi}{6}\right)$.

We know that the range of the principal value of

$\cot^{-1}x$ is $(0, \pi)$ and $\cot\left(\frac{\pi}{6}\right) = \sqrt{3}$.

Hence, the principal value of $\cot^{-1}\sqrt{3}$ is $\frac{\pi}{6}$.

9. Find the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Solution:

Let $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = y$, then

$\cos y = -\frac{1}{\sqrt{2}} = -\cos\left(\frac{\pi}{4}\right) = \cos\left(\pi - \frac{\pi}{4}\right) = \cos\left(\frac{3\pi}{4}\right).$

We know that the range of the principal value of

$\cos^{-1}x$ is $[0, \pi]$ and $\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}.$

Hence, the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ is $\frac{3\pi}{4}$.

10. Find the principal value of $\text{cosec}^{-1}(-\sqrt{2})$

Solution:

Let $cosec^{-1}(-\sqrt{2}) = y$, then

$cosec\, y = -\sqrt{2} = -cosec\left(\frac{\pi}{4}\right) = cosec\left(-\frac{\pi}{4}\right)$

We know that the range of the principal value of

$\text{cosec}^{-1}x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ and $\text{cosec}\left(-\frac{\pi}{4}\right) = -\sqrt{2}.$

Hence, the principal value of $\text{cosec}^{-1}(-\sqrt{2})$ is $-\frac{\pi}{4}$.

11. Find the value of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$

Solution:

Let $tan^{-1}(1) = x$, then $\tan x = 1 = tan\frac{\pi}{4}$

We know that the range of the principal value of $\tan^{-1}x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $tan\frac{\pi}{4} =$ 1.

$\therefore tan^{-1}(1) = \frac{\pi}{4}$

Let $cos^{-1}\left(-\frac{1}{2}\right) = \text{y}$, then

$\cos \text{y} = -\frac{1}{2} = -cos\frac{\pi}{3} = cos\left(\pi - \frac{\pi}{3}\right) = cos\left(\frac{2\pi}{3}\right)$

We know that the range of the principal value of $\cos^{-1}x$ is $[0, \pi]$ and $cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

$\therefore cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

Let $sin^{-1}\left(-\frac{1}{2}\right) =$ z, then

$\sin z = -\frac{1}{2} = -sin\frac{\pi}{6} = sin\left(-\frac{\pi}{6}\right)$

We know that the range of the principal value of $\sin^{-1}x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

$\therefore \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

Now,

$\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$

$= \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6} = \frac{3\pi+8\pi-2\pi}{12} = \frac{9\pi}{12} = \frac{3\pi}{4}$

12. Find the value of $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$

Solution:

Let $\cos^{-1}\left(\frac{1}{2}\right) = x$, then

$\cos x = \frac{1}{2} = \cos\frac{\pi}{3}$

We know that the range of the principal value of $\cos^{-1}x$ is $[0, \pi]$ and $\cos\frac{\pi}{3} = \frac{1}{2}$.

Hence, $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

Let $\sin^{-1}\left(\frac{1}{2}\right) = y$, then

$\sin y = \frac{1}{2} = \sin\frac{\pi}{6}$

We know that the range of the principal value of

$\sin^{-1}x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin\frac{\pi}{6} = \frac{1}{2}$

$\therefore sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

Now,

$$cos^{-1}\left(\frac{1}{2}\right) + 2sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} + 2 \times \frac{\pi}{6} = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$$

13. If $\sin^{-1} x = y$, then

(A) $0 \le y \le \pi$

(B) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(C) $0 < y < \pi$

(D) $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Solution:

It is given that $\sin^{-1} x = y$.

We know that the range of the principal value of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Hence, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Hence, the option (B) is correct.

14. $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$ is equal to

(A) π

(B) $-\frac{\pi}{3}$

(C) $\frac{\pi}{3}$

(D) $\frac{2\pi}{3}$

Solution:

Let $\tan^{-1}\sqrt{3} = x$, then

$\tan x = \sqrt{3} = \tan\frac{\pi}{3}$

We know that the range of the principal value of $\tan^{-1}x$ is $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

$\therefore \tan^{-1}\sqrt{3}=\frac{\pi}{3}$

Let $\sec^{1}(-2)=y$, then

$\sec y=-2=-\sec\frac{\pi}{2}=\sec\left(\pi-\frac{\pi}{3}\right)=\sec\left(\frac{2\pi}{3}\right)$

We know that the range of the principal value of $\sec^{-1}x$ is $[0,\pi]-\left\{\frac{\pi}{2}\right\}$

$\therefore \sec^{-1}(-2)=\frac{2\pi}{3}$

Now,

$tan^{-1}\sqrt{3}-sec^{-1}(-2)=\frac{\pi}{3}-\frac{2\pi}{3}=-\frac{\pi}{3}$

Hence, the option (B) is correct.

EXERCISE 2.2

1. Prove that $3\sin^{-1}x=\sin^{-1}(3x-4x^3), x\in\left[-\frac{1}{2},\frac{1}{2}\right]$

Solution:

Let $sin^{-1}x=\theta$, then $x=\sin\theta$

Since, $x\in\left[-\frac{1}{2},\frac{1}{2}\right]$

Hence, $\theta\in\left[-\frac{\pi}{6},\frac{\pi}{6}\right]$

Now,

RHS $=sin^{-1}(3x-4x^3)=sin^{-1}(3\sin\theta-4sin^3\theta)$

$=\sin^{-1}(sin\,3\theta)$

$=3\theta$ (Since, $3\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$)

$=3sin^{-1}x=$ LHS

Thus, LHS = RHS

2. Prove that $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x), x \in \left[\frac{1}{2}, 1\right]$

Solution:

Let $cos^{-1}x = \theta$, then $x = \cos\theta$

Since, $x \in \left[\frac{1}{2}, 1\right]$

Hence, $\theta \in \left[0, \frac{\pi}{3}\right]$

Now,

RHS $= cos^{-1}(4x^3 - 3x) = cos^{-1}(4cos^3\theta - 3cos\theta)$

$= cos^{-1}(\cos 3\theta)$

$= 3\theta$ (Since, $3\theta \in [0, \pi]$)

$= 3cos^{-1}x =$ LHS

Thus, LHS = RHS

3. Prove that $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$

Solution:

As we know that when $xy < 1$, $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$

Here, $x = \frac{2}{11}, y = \frac{7}{24}$. Hence, $xy = \frac{7}{132} < 1$

So, LHS $= \tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24}$

$= tan^{-1}\left(\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}\times\frac{7}{24}}\right)$

$= tan^{-1}\left(\frac{\frac{48+77}{11\times24}}{\frac{11\times24-14}{11\times24}}\right)$

$= tan^{-1}\frac{48+77}{264-14} = tan^{-1}\frac{125}{250} = tan^{-1}\frac{1}{2} =$ RHS

Thus, LHS = RHS

4. Prove that $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{31}{17}$

Solution:

As we know that when $|x|<1, 2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}$ and when $xy<1$, $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}$

So, LHS $=2tan^{-1}\frac{1}{2}+tan^{-1}\frac{1}{7}$

$=tan^{-1}\left[\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}\right]+tan^{-1}\frac{1}{7}$ (Since, $\left|\frac{1}{2}\right|<1$)

$=tan^{-1}\frac{1}{\left(\frac{3}{4}\right)}+tan^{-1}\frac{1}{7}$

$=tan^{-1}\frac{4}{3}+tan^{-1}\frac{1}{7}$

$=tan^{-1}\left(\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\times\frac{1}{7}}\right)$ (Since, $\frac{4}{3}\times\frac{1}{7}=\frac{4}{21}<1$)

$=tan^{-1}\left(\frac{\frac{28+3}{3\times7}}{\frac{3\times7-4}{3\times7}}\right)=tan^{-1}\frac{28+3}{21-4}=tan^{-1}\frac{31}{17}=$ RHS

Thus, LHS = RHS.

5. Write $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}, x\neq0$ in simplest form.

Solution:

Given expression is $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$

Let $x=\tan\theta$. Hence $\theta=\tan^{-1}x$

$\therefore tan^{-1}\frac{\sqrt{1+x^2}-1}{x}=\tan^{-1}\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}$

$=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$

$= \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$

$= \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$

6. Write $\tan^{-1}\frac{1}{\sqrt{x^2-1}}, |x| > 1$ in simplest form.

Solution:

Given expression is $\tan^{-1}\frac{1}{\sqrt{x^2-1}}$

Let $x = \operatorname{cosec}\theta$. Hence $\theta = \operatorname{cosec}^{-1}x$

$\therefore \tan^{-1}\frac{1}{\sqrt{x^2-1}} = \tan^{-1}\frac{1}{\sqrt{cosec^2\,\theta-1}}$

$= \tan^{-1}\frac{1}{\cot\theta} = \tan^{-1}\tan\theta = \theta = cosec^{-1}\,x$

$= \frac{\pi}{2} - \sec^{-1}x$ (Since, $cosec^{-1}\,x + sec^{-1}\,x = \frac{\pi}{2}$)

7. Write $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right), 0 < x < \pi$ in simplest form.

Solution:

The given expression is $tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$,

Now,

$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right) = \tan^{-1}\left(\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right)$

$= \tan^{-1}\left(\sqrt{\tan^2\frac{x}{2}}\right)$

$= \tan^{-1}\left(\tan\frac{x}{2}\right)$ (Since, $0 < \frac{x}{2} < \frac{\pi}{2}$. Hence, $\tan\frac{x}{2} > 0$)

$= \frac{x}{2}$ (Since, $\tan^{-1}(\tan x) = x, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$)

8. Write $\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right), \frac{-\pi}{4}<x<\frac{3\pi}{4}$ in simplest form.

Solution:

The given expression is $tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$

Now,

$tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)=tan^{-1}\left(\frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}\right)=tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right)$

$=tan^{-1}\left(\frac{1-\tan x}{1+1.\tan x}\right)=tan^{-1}\left(\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}.\tan x}\right)$

$=tan^{-1}\left[tan\left(\frac{\pi}{4}-x\right)\right]$

Since, $\frac{-\pi}{4}<x<\frac{3\pi}{4}$

$\Rightarrow\frac{-3\pi}{4}<-x<\frac{\pi}{4}$

$\Rightarrow\frac{\pi}{4}-\frac{3\pi}{4}<\frac{\pi}{4}-x<\frac{\pi}{4}+\frac{\pi}{4}$

$\Rightarrow-\frac{\pi}{2}<\frac{\pi}{4}-x<\frac{\pi}{2}$

Hence, $tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)=\frac{\pi}{4}-x$ (Since, $\tan^{-1}(\tan x)=x, x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$)

9. Write $\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}, |x|<a$ in simplest form.

Solution:

The given expression is $\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$.

Let $x=a\sin\theta$. Hence, $\theta=\sin^{-1}\frac{x}{a}$

$\therefore\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}=\tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right)=\tan^{-1}\left(\frac{a\sin\theta}{a\sqrt{1-\sin^2\theta}}\right)$

$=\tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right)=\tan^{-1}(\tan\theta)=\theta=\sin^{-1}\frac{x}{a}$

10. Write $\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right), a>0; \frac{-a}{\sqrt{3}}<x<\frac{a}{\sqrt{3}}$ in simplest form.

Solution:

The given expression is $tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)$

Let $x = a\tan\theta$. Hence, $\theta = \tan^{-1}\frac{x}{a}$

$\therefore \tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right) = \tan^{-1}\left(\frac{3a^2.a\tan\theta - a^3\tan^3\theta}{a^3-3a.a^2\tan^2\theta}\right)$

$$= \tan^{-1}\left(\frac{3a^3\,tan\,\theta - a^3\,tan^3\,\theta}{a^3-3a^3\,tan^2\,\theta}\right)$$

$$= \tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$$

$= tan^{-1}(\tan 3\theta)$

Since, $\frac{-a}{\sqrt{3}}<x<\frac{a}{\sqrt{3}}$

$\Rightarrow \frac{-a}{\sqrt{3}}<a\tan\theta<\frac{a}{\sqrt{3}}$

$\Rightarrow \frac{-1}{\sqrt{3}}<\tan\theta<\frac{1}{\sqrt{3}}$

$\Rightarrow -\frac{\pi}{6}<\theta<\frac{\pi}{6}$

$\Rightarrow -\frac{\pi}{2}<3\theta<\frac{\pi}{2}$

Hence, $tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)$

$= 3\theta = 3\tan^{-1}\frac{x}{a}$ (Since, $\tan^{-1}(\tan x) = x, x \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$)

11. Find the value of $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$.

Solution:

The given expression is $tan^{-1}\left[2cos\left(2sin^{-1}\frac{1}{2}\right)\right]$

$\therefore tan^{-1}\left[2cos\left(2sin^{-1}\frac{1}{2}\right)\right]$

$= tan^{-1}\left[2\,cos\left(2sin^{-1}\left(sin\,\frac{\pi}{6}\right)\right)\right]$

$= tan^{-1}\left[2cos\left(2\times\frac{\pi}{6}\right)\right]$ (Since, $\sin^{-1}(sinx) = x, x \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$)

$= tan^{-1}\left[2cos\left(\frac{\pi}{3}\right)\right]$

$= tan^{-1}\left[2\times\frac{1}{2}\right]$

$= tan^{-1}[1] = \frac{\pi}{4}$

12. Find the value of $\cot(\tan^{-1}a + \cot^{-1}a)$?

Solution:

The given expression is $cot(tan^{-1}a + cot^{-1}a)$.

we know that, $tan^{-1}x + cot^{-1}x = \frac{\pi}{2}$ ...(1)

Substituting equation (1) in the given expression

$\therefore cot(tan^{-1}a + cot^{-1}a) = cot\left(\frac{\pi}{2}\right) = 0$

13. Find the value of $\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right], |x| < 1, y > 0$ and $xy < 1$?

Solution:

The given expression is $tan\,\frac{1}{2}\left[sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right]$

$= tan\,\frac{1}{2}\left[sin^{-1}\frac{2x}{1+x^2} + cos^{-1}\frac{1-y^2}{1+y^2}\right]$

$= tan\frac{1}{2}[2tan^{-1}x + 2tan^{-1}y]$

$\left[we\ know\ that, 2tan^{-1}x = sin^{-1}\frac{2x}{1+x^2} = cos^{-1}\frac{1-x^2}{1+x^2}\right]$

$= tan\frac{1}{2}[2(tan^{-1}x + tan^{-1}y)]$

$= \tan[tan^{-1}x + tan^{-1}y]$

$= tan\left[tan^{-1}\frac{x+y}{1-xy}\right] = \frac{x+y}{1-xy}$

14. f $\sin(\sin^{-1} 5 + \cos^{-1} x) = 1$, then find the value of x?

Solution:

Since, $\sin\left(sin^{-1}\frac{1}{5} + cos^{-1}x\right) = 1$

$\therefore \left(sin^{-1}\frac{1}{5} + cos^{-1}x\right) = sin^{-1}1$

$\Rightarrow \left(sin^{-1}\frac{1}{5} + cos^{-1}x\right) = \frac{\pi}{2}$

$\Rightarrow sin^{-1}\frac{1}{5} = \frac{\pi}{2} - cos^{-1}x$

$\Rightarrow sin^{-1}\frac{1}{5} = sin^{-1}x \left[as \sin^{-1}x + cos^{-1}x = \frac{\pi}{2}\right]$

$\Rightarrow x = \frac{1}{5}$

15. If $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$, then find the value of x?

Solution:

Given that $tan^{-1}\frac{x-1}{x-2} + tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$

$\Rightarrow tan^{-1}\left(\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\frac{x-1}{x-2}x\frac{x+1}{x+2}}\right) = \frac{\pi}{4}\left[as \tan^{-1}x + tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$\Rightarrow \frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\frac{x-1}{x-2}\times\frac{x+1}{x+2}} = tan\frac{\pi}{4}$

$\Rightarrow \frac{\left[\frac{(x-1)(x+2)+(x-2)(x+1)}{(x-2)(x+2)}\right]}{\left[\frac{(x-2)(x+2)-(x-1)(x+1)}{(x-2)(x+2)}\right]} = 1$

$\Rightarrow \frac{x^2+2x-x-2+x^2+x-2x-2}{x^2-4-(x^2-1)} = 1$

$\Rightarrow \frac{2x^2-4}{-3} = 1$

$\Rightarrow 2x^2 - 4 = -3 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}.$

16. Find the value of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$?

Solution:

Given that $sin^{-1}\left(\sin\frac{2\pi}{3}\right)$.

We know that $\sin^{-1}(\sin x) = x$ if $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

Hence, $sin^{-1}\left(\sin\frac{2\pi}{3}\right) = sin^{-1}\left(\sin\left\{\pi - \frac{\pi}{3}\right\}\right)$

$= \sin^{-1}\left(sin\frac{\pi}{3}\right) = \frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Hence, $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \frac{\pi}{3}$

17. Find the value of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$?

Solution:

Given that $tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

We know that $\tan^{-1}(\tan x) = x$ if $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$\therefore tan^{-1}\left(\tan\frac{3\pi}{4}\right) = tan^{-1}\left(\tan\left\{\pi - \frac{\pi}{4}\right\}\right)$

$= tan^{-1}\left(-\tan\frac{\pi}{4}\right)$

$= tan^{-1}\left(\tan\left\{-\frac{\pi}{4}\right\}\right)$

$= -\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Hence, $tan^{-1}\left(\tan\frac{3\pi}{4}\right) = -\frac{\pi}{4}$

18. Find the value of $\tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$?

Solution:

Given that $tan\left(sin^{-1}\frac{3}{5}+cot^{-1}\frac{3}{2}\right)$

$\therefore tan\left(sin^{-1}\frac{3}{5}+cot^{-1}\frac{3}{2}\right)=tan\left(tan^{-1}\frac{3}{\sqrt{5^2-3^2}}+tan^{-1}\frac{2}{3}\right)$

$\left[as\sin^{-1}\frac{a}{b}=tan^{-1}\frac{a}{\sqrt{b^2-a^2}}\ and\ \cot^{-1}\frac{a}{b}=tan^{-1}\frac{b}{a}\right]$

$=tan\left(tan^{-1}\frac{3}{4}+tan^{-1}\frac{2}{3}\right)$

$=\tan\left[\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}\right)\right]$

$=\tan\left[\tan^{-1}\left(\frac{\frac{9+8}{4\times3}}{\frac{4\times3-3\times2}{4\times3}}\right)\right]$

$=\tan\left(\tan^{-1}\frac{17}{6}\right)=\frac{17}{6}$

19. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ is equal to

(A) $\frac{7\pi}{6}$

(B) $\frac{5\pi}{6}$

(C) $\frac{\pi}{3}$

(D) $\frac{\pi}{6}$

Solution:

Given that $cos^{-1}\left(cos\frac{7\pi}{6}\right)$

We know that $\cos^{-1}(\cos x)=x$, if $x\in[0,\pi]$,

$\therefore cos^{-1}\left(cos\frac{7\pi}{6}\right)$

$=\cos^{-1}\left[\left(cos\ (2\pi-\frac{5\pi}{6}\right)\right]$

$= cos^{-1}\left(cos\frac{5\pi}{6}\right) = \frac{5\pi}{6} \in [0, \pi]$

Hence, $cos^{-1}\left(cos\frac{7\pi}{6}\right) = \frac{5\pi}{6}$

Hence, the option (B) is correct.

20. $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$ is equal to

(A) $\frac{1}{2}$

(B) $\frac{1}{3}$

(C) $\frac{1}{4}$

(D) 1

Solution:

Given that $sin\left(\frac{\pi}{3} - sin^{-1}\left(-\frac{1}{2}\right)\right)$

We know that the range of the principal value of $\sin^{-1}x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$\therefore sin\left(\frac{\pi}{3} - sin^{-1}\left(-\frac{1}{2}\right)\right)$

$= \sin\left[\frac{\pi}{3} - sin^{-1}\left(-sin\frac{\pi}{6}\right)\right]$

$= sin\left[\frac{\pi}{3} - sin^{-1}\left\{sin\left(-\frac{\pi}{6}\right)\right\}\right]$

$= sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right)$

$= sin\left(\frac{3\pi}{6}\right)$

$= sin\frac{\pi}{2} = 1$

Hence, $sin\left(\frac{\pi}{3} - sin^{-1}\left(-\frac{1}{2}\right)\right) = 1$

Hence, the option (D) is correct.

21. $\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$ is equal to

(A) π

(B) $-\frac{\pi}{2}$

(C) 0

(D) $2\sqrt{3}$

Solution:

Given that $tan^{-1}\sqrt{3} - cot^{-1}(-\sqrt{3})$

We know that the range of the principal value of $\tan^{-1}x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\cot^{-1}x$ is $(0, \pi)$.

$\therefore tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$

$= tan^{-1}\left(\tan\frac{\pi}{3}\right) - \cot^{-1}\left(-\cot\frac{\pi}{6}\right)$

$= \frac{\pi}{3} - cot^{-1}\left[cot\left(\pi - \frac{\pi}{6}\right)\right]$ (Since, $\tan^{-1}(tanx) = x, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$)

$= \frac{\pi}{3} - cot^{-1}\left(cot\frac{5\pi}{6}\right)$

$= \frac{\pi}{3} - \frac{5\pi}{6}$ (Since, $\cot^{-1}(cotx) = x, x \in (0, \pi)$)

$= \frac{2\pi - 5\pi}{6}$

$= \frac{-3\pi}{6}$

$= -\frac{\pi}{2}$

Hence, $tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3}) = -\frac{\pi}{2}$

Hence, the options (B) is correct.

Miscellaneous Exercise on Chapter 2

1. Find the value of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$

Solution:

Given that $cos^{-1}\left(cos\frac{13\pi}{6}\right)$

We know that $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi]$,

$\therefore cos^{-1}\left(cos\frac{13\pi}{6}\right)$

$= cos^{-1}\left[cos\left(2\pi + \frac{\pi}{6}\right)\right]$

$= cos^{-1}\left(cos\frac{\pi}{6}\right)$

$= \frac{\pi}{6} \in [0, \pi]$

Hence, $cos^{-1}\left(cos\frac{13\pi}{6}\right) = \frac{\pi}{6}$

2. Find the value of $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

Solution:

Given that $tan^{-1}\left(tan\ \frac{7\pi}{6}\right)$

We know that $\tan^{-}(tan\ x) = x$ if, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$\therefore tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

$= tan^{-1}\left[tan\left(\pi + \frac{\pi}{6}\right)\right]$

$= tan^{-1}\left(tan\frac{\pi}{6}\right) = \frac{\pi}{6}$

Hence, $tan^{-1}\left(tan\frac{7\pi}{6}\right) = \frac{\pi}{6}$

3. Prove that, $2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$

Solution:

LHS = $2sin^{-1}\frac{3}{5}$

$= 2tan^{-1}\frac{3}{\sqrt{5^2-3^2}}\left[as\sin^{-1}\frac{a}{b} = tan^{-1}\frac{a}{\sqrt{b^2-a^2}}\right]$

$= 2tan^{-1}\frac{3}{4} = tan^{-1}\left[\frac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2}\right]\left[as\ 2tan^{-1}x = tan^{-1}\frac{2x}{1-x^2}\right]$

$= tan^{-1}\left[\frac{\frac{3}{2}}{\frac{16-9}{16}}\right]$

$= tan^{-1}\left(\frac{3}{2}\times\frac{16}{7}\right)$

$= tan^{-1}\frac{24}{7}$ =RHS

Hence Proved, RHS = LHS

4. Prove that, $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$

Solution:

LHS = $sin^{-1}\frac{8}{17} + sin^{-1}\frac{3}{5}$

$= tan^{-1}\frac{8}{\sqrt{17^2-8^2}} + tan^{-1}\frac{3}{\sqrt{5^2-3^2}}\left[as\sin^{-1}\frac{a}{b} = tan^{-1}\frac{a}{\sqrt{b^2-a^2}}\right]$

$= tan^{-1}\frac{8}{15} + \tan^{-1}\frac{3}{4}$

$= tan^{-1}\left[\frac{\frac{8}{15}+\frac{3}{4}}{1-\frac{8}{15}\times\frac{3}{4}}\right]\left[as\tan^{-1}x + tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$= tan^{-1}\left[\frac{\frac{32+45}{15\times 4}}{\frac{15\times 4-8\times 3}{15\times 4}}\right]$

$= tan^{-1}\left[\frac{\frac{77}{60}}{\frac{36}{60}}\right]$

$= tan^{-1}\frac{77}{36}$ =RHS

Hence Proved, RHS = LHS

5. Prove that, $\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}=\cos^{-1}\frac{33}{65}$

Solution:

LHS= $\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}$

$=\tan^{-1}\frac{\sqrt{5^2-4^2}}{4}+\tan^{-1}\frac{\sqrt{13^2-12^2}}{12}$ $\left[as\cos^{-1}\frac{a}{b}=\tan^{-1}\frac{\sqrt{b^2-a^2}}{a}\right]$

$=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{5}{12}$

$=\tan^{-1}\left[\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\times\frac{5}{12}}\right]$ $\left[as\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$=\tan^{-1}\left[\frac{\frac{36+20}{4\times 12}}{\frac{4\times 12-3\times 5}{4\times 12}}\right]=\tan^{-1}\frac{56}{33}$

$=\cos^{-1}\frac{33}{\sqrt{56^2+33^2}}$ $\left[as\tan^{-1}\frac{a}{b}=\cos^{-1}\frac{b}{\sqrt{a^2+b^2}}\right]$

$=\cos^{-1}\frac{33}{\sqrt{4225}}=\cos^{-1}\frac{33}{65}=$ RHS

Hence Proved, RHS = LHS

6. Prove that, $\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{56}{65}$

Solution:

LHS = $\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}$

$=\tan^{-1}\frac{\sqrt{13^2-12^2}}{12}+\tan^{-1}\frac{3}{\sqrt{5^2-3^2}}$

$\left[as\cos^{-1}\frac{a}{b}=\tan^{-1}\frac{\sqrt{b^2-a^2}}{a}\ and\ \sin^{-1}\frac{a}{b}=\tan^{-1}\frac{a}{\sqrt{b^2-a^2}}\right]$

$=\tan^{-1}\frac{5}{12}+\tan^{-1}\frac{3}{4}$

$=\tan^{-1}\left[\frac{\frac{5}{12}+\frac{3}{4}}{1-\frac{5}{12}\times\frac{3}{4}}\right]$ $\left[as\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$= \tan^{-1}\left[\dfrac{\frac{20+36}{12\times 4}}{\frac{12\times 4-5\times 3}{12\times 4}}\right] = \tan^{-1}\dfrac{56}{33}$

$= \sin^{-1}\frac{56}{\sqrt{56^2+33^2}}$ $\left[as \tan^{-1}\frac{a}{b} = \sin^{-1}\frac{a}{\sqrt{a^2+b^2}}\right]$

$= \sin^{-1}\frac{56}{\sqrt{4225}} = \sin^{-1}\frac{56}{65}$ =RHS

Hence Proved, RHS = LHS

7. Prove that, $\tan^{-1}\frac{63}{16} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$

Solution:

RHS $= sin^{-1}\frac{5}{13} + cos^{-1}\frac{3}{5}$

$= tan^{-1}\frac{5}{\sqrt{13^2-5^2}} + tan^{-1}\frac{\sqrt{5^2-3^2}}{3}$

$\left[as \cos^{-1}\frac{a}{b} = tan^{-1}\frac{\sqrt{b^2-a^2}}{a}\ and\ sin^{-1}\frac{a}{b} = tan^{-1}\frac{a}{\sqrt{b^2-a^2}}\right]$

$= tan^{-1}\dfrac{5}{12} + \tan^{-1}\dfrac{4}{3}$

$= tan^{-1}\left[\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\times\frac{4}{3}}\right]$ $\left[as \tan^{-1}x + tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$= tan^{-1}\left[\frac{\frac{15+48}{12\times 3}}{\frac{12\times 3-5\times 4}{12\times 3}}\right]$

$= tan^{-1}\frac{63}{16}$ =LHS

Hence Proved, LHS = RHS

8. Prove that, $\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

Solution:

LHS $= tan^{-1}\frac{1}{5} + tan^{-1}\frac{1}{7} + tan^{-1}\frac{1}{3} + tan^{-1}\frac{1}{8}$

$= tan^{-1}\left[\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right] + tan^{-1}\left[\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right]$

$\left[as\ tan^{-1}x + tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$= tan^{-1}\left[\frac{\frac{7+5}{5\times7}}{\frac{5\times7-1\times1}{5\times7}}\right] + tan^{-1}\left[\frac{\frac{8+3}{3\times8}}{\frac{3\times8-1\times1}{3\times8}}\right]$

$= tan^{-1}\frac{12}{34} + tan^{-1}\frac{11}{23} = tan^{-1}\frac{6}{17} + tan^{-1}\frac{11}{23}$

$= tan^{-1}\left[\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right]\left[as\ \tan^{-1}x + tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$= tan^{-1}\left[\frac{\frac{138+187}{17\times23}}{\frac{17\times23-6\times11}{17\times23}}\right] = tan^{-1}\left(\frac{138+187}{391-66}\right)$

$= tan^{-1}\frac{325}{3Z5} = tan^{-1}1 = \frac{\pi}{4}$ =RHS

Hence Proved, RHS = LHS

9. Prove that, $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\frac{1-x}{1+x}, x \in [0,1]$

Solution:

Given equation, $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\frac{1-x}{1+x}, x \in [0,1]$

LHS = $tan^{-1}\sqrt{x}$

$= \frac{1}{2}\times 2tan^{-1}\sqrt{x}$

$= \frac{1}{2}cos^{-1}\left[\frac{1-(\sqrt{x})^2}{1+(\sqrt{x})^2}\right]\left[as\ 2tan^{-1}x = cos^{-1}\left[\frac{1-x^2}{1+x^2}\right], x \geq 0\right]$

$= \frac{1}{2}cos^{-1}\left(\frac{1-x}{1+x}\right)$ =RHS

Hence Proved, RHS = LHS

10. Prove that, $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}, x \in \left(0,\frac{\pi}{4}\right)$

Solution:

LHS $= \cos^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$

$= \cot^{-1}\left(\frac{\sqrt{1+\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{1-\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{1+\cos\left(\frac{\pi}{2}-x\right)}-\sqrt{1-\cos\left(\frac{\pi}{2}-x\right)}}\right)$

$= \cot^{-1}\left(\frac{\sqrt{1+\cos y}+\sqrt{1-\cos y}}{\sqrt{1+\cos y}-\sqrt{1-\cos y}}\right)$ $\left[Let\, \frac{\pi}{2} - x = y\right]$

$= \cot^{-1}\left(\frac{\sqrt{2\cos^2\frac{y}{2}}+\sqrt{2\sin^2\frac{y}{2}}}{\sqrt{2\cos^2\frac{y}{2}}-\sqrt{2\sin^2\frac{y}{2}}}\right)$

$\left[as\ 1 + \cos y = 2\cos^2\frac{y}{2}\ and\ 1 - \cos y = 2\sin^2\frac{y}{2}\right]$

$= \cot^{-1}\left(\frac{\sqrt{2}\cos\frac{y}{2}+\sqrt{2}\sin\frac{y}{2}}{\sqrt{2}\cos\frac{y}{2}-\sqrt{2}\sin\frac{y}{2}}\right)$

$= \cot^{-1}\left(\frac{1+\tan\frac{y}{2}}{1-\tan\frac{y}{2}}\right)$ $\left[Dividing\ each\ term\ by\ \sqrt{2}\cos\frac{y}{2}\right]$

$= \cot^{-1}\left(\frac{\tan\frac{\pi}{4}+\tan\frac{y}{2}}{1-\tan\frac{\pi}{4}.\tan\frac{y}{2}}\right)$

$= \cot^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{y}{2}\right)\right]$

$= \cot^{-1}\left[\cot\left\{\frac{\pi}{2}-\left(\frac{\pi}{4}+\frac{y}{2}\right)\right\}\right]$

$= \frac{\pi}{2}-\left(\frac{\pi}{4}+\frac{y}{2}\right) = \frac{\pi}{4}-\frac{y}{2}$

$= \frac{\pi}{4}-\frac{1}{2}\left(\frac{\pi}{2}-x\right)$

$= \frac{x}{2}$ =RHS

Hence Proved, LHS = RHS

11. Prove that, $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x, -\frac{1}{\sqrt{2}}\leq x\leq 1$ [Hint: Put $x=\cos 2\theta$]

Solution:

LHS= $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

$=\tan^{-1}\left(\frac{\sqrt{1+\cos y}-\sqrt{1-\cos y}}{\sqrt{1+\cos y}+\sqrt{1-\cos y}}\right)$ $\left[Let\ x=\cos y, y\in\left[0,\frac{3\pi}{4}\right]\right]$

$=\tan^{-1}\left(\frac{\sqrt{2\cos^2\frac{y}{2}}-\sqrt{2\sin^2\frac{y}{2}}}{\sqrt{2\cos^2\frac{y}{2}}+\sqrt{2\sin^2\frac{y}{2}}}\right)$

$\left[as\ 1+\cos y=2\cos^2\frac{y}{2}\ and\ 1-\cos y=2\sin^2\frac{y}{2}\right]$

$=\tan^{-1}\left(\frac{\sqrt{2}\cos\frac{y}{2}-\sqrt{2}\sin\frac{y}{2}}{\sqrt{2}\cos\frac{y}{2}+\sqrt{2}\sin\frac{y}{2}}\right)$ [Since, $\frac{y}{2}\in\left[0,\frac{3\pi}{8}\right]$, hence $\cos\frac{y}{2}$ and $\sin\frac{y}{2}$ are positive.]

$=\tan^{-1}\left(\frac{1-\tan\frac{y}{2}}{1+\tan\frac{y}{2}}\right)$ $\left[Dividing\ each\ term\ by\ \sqrt{2}\cos\frac{y}{2}\right]$

$=\tan^{-1}\left(\frac{\tan\frac{\pi}{4}-\tan\frac{y}{2}}{1+\tan\frac{\pi}{4}.\tan\frac{y}{2}}\right)$

$=\tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{y}{2}\right)\right]$

$=\frac{\pi}{4}-\frac{y}{2}$ $\left[\frac{\pi}{4}-\frac{y}{2}\in\left[-\frac{\pi}{8},\frac{\pi}{4}\right]\right]$

$=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x$ =RHS

Hence Proved, RHS = LHS

12. Prove that, $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}=\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

Solution:

LHS $=\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}=\frac{9}{4}\left(\frac{\pi}{2}-\sin^{-1}\frac{1}{3}\right)$

$= \frac{9}{4}\left(\cos^{-1}\frac{1}{3}\right)$ $\left[as \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\right]$

$= \frac{9}{4}\left(\sin^{-1}\frac{\sqrt{3^2-1^2}}{3}\right)$ $\left[as \cos^{-1}\frac{a}{b} = \sin^{-1}\frac{\sqrt{b^2-a^2}}{b}\right]$

$= \frac{9}{4}\left(\sin^{-1}\frac{\sqrt{8}}{3}\right)$

$= \frac{9}{4}\left(\sin^{-1}\frac{2\sqrt{2}}{3}\right)$ =RHS

Hence Proved, LHS = RHS

13. solve $2\tan^{-1}(\cos x) = \tan^{-1}(2 \text{ cosec } x)$

Solution:

Given: $2\tan^{-1}(\cos x) = \tan^{-1}(2\, cosec\, x)$

$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\, cosec\, x)$ $\left[as\ 2\tan^{-1} x = \tan^{-1}\frac{2x}{1-x^2}\right]$

$\Rightarrow \frac{2\cos x}{1-\cos^2 x} = 2\, cosec\, x$

$\Rightarrow \frac{2\cos x}{\sin^2 x} = \frac{2}{\sin x}$

$\Rightarrow 2\sin x . \cos x = 2\sin^2 x$

$\Rightarrow 2\sin x . \cos x - 2\sin^2 x = 0 \Rightarrow 2\sin x(\cos x - \sin x) = 0$

$\Rightarrow 2\sin x = 0$ or $\cos x - \sin x = 0$

But $\sin x \neq 0$ as it does not satisfy the equation

$\therefore \cos x - \sin x = 0 \Rightarrow \cos x = \sin x \Rightarrow \tan x = 1$

$\therefore x = \frac{\pi}{4}$

14. Solve $\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1} x, (x > 0)$

Solution:

Given that $tan^{-1}\frac{1-x}{1+x}=\frac{1}{2}tan^{-1}x$

$\Rightarrow tan^{-1}1-tan^{-1}x=\frac{1}{2}tan^{-1}x$

$\left[\therefore \tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$

$\Rightarrow \frac{\pi}{4}=\frac{3}{2}\,tan^{-1}x \Rightarrow \frac{\pi}{6}=tan^{-1}x$

$\Rightarrow tan\left(\frac{\pi}{6}\right)=x$

$\therefore x=\frac{1}{\sqrt{3}}$

15. $\sin(\tan^{-1}x), |x|<1$ is equal to

(A) $\frac{x}{\sqrt{1-x^2}}$

(B) $\frac{1}{\sqrt{1-x^2}}$

(C) $\frac{1}{\sqrt{1+x^2}}$

(D) $\frac{x}{\sqrt{1+x^2}}$

Solution:

Given that: $sin(tan^{-1}x)$

$=\sin\left(sin^{-1}\frac{x}{\sqrt{1+x^2}}\right)\quad\left[as\tan^{-1}\frac{a}{b}=\sin^{-1}\frac{a}{\sqrt{a^2+b^2}}\right]$

$=\frac{x}{\sqrt{1+x^2}}$

Hence, the option (D) is correct.

16. $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$, then x is equal to

(A) $0,\frac{1}{2}$

(B) $1,\frac{1}{2}$

(C) 0

(D) $\frac{1}{2}$

Solution:

Given that $sin^{-1}(1-x)-2sin^{-1}x=\frac{\pi}{2}$

Let $x=\sin y$, hence $y=\sin^{-1}x$

$\therefore sin^{-1}(1-\sin y)-2y=\frac{\pi}{2}$

$\Rightarrow sin^{-1}(1-\sin y)=\frac{\pi}{2}+2y$

$\Rightarrow 1-\sin y=sin\left(\frac{\pi}{2}+2y\right)$

$\Rightarrow 1-\sin y=\cos 2y$

$\Rightarrow 1-\sin y=1-2sin^2y\ \ [as\ \cos 2y=1-2sin^2y]$

$\Rightarrow 2sin^2y-siny=0$

$\Rightarrow 2x^2-x=0\ [as\ x\ =sin\ y]$

$\Rightarrow x(2x-1)=0$

$\Rightarrow x=0$ or $x=\frac{1}{2}$

But $x\neq\frac{1}{2}$, as it does not satisfy the given equation.

$\therefore x=0$ is the solution of the given equation.

Hence, the option (C) is correct.

17. The value of $\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\frac{x-y}{x+y}$ is equal to

(A) $\frac{\pi}{2}$

(B) $\frac{\pi}{3}$

(C) $\frac{\pi}{4}$

(D) $-\frac{3\pi}{4}$

Solution:

$$tan^{-1}\left(\frac{x}{y}\right) - tan^{-1}\frac{x-y}{x+y}$$

$$= tan^{-1}\left[\frac{\frac{x}{y}-\frac{x-y}{x+y}}{1+\frac{x}{y}\times\frac{x-y}{x+y}}\right] \left[as \tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right.$$

$$= tan^{-1}\left[\frac{\frac{x(x+y)-y(x-y)}{y(x+y)}}{\frac{y(x+y)+x(x-y)}{y(x+y)}}\right]$$

$$= tan^{-1}\left[\frac{x^2+xy-xy+y^2}{xy+y^2+x^2-xy}\right]$$

$$= tan^{-1}\left[\frac{x^2+y^2}{x^2+y^2}\right]$$

$$= tan^{-1}1 = \frac{\pi}{4}$$

Hence, the option (C) is correct.