

CBSE NCERT Solutions for Class 12 Maths Chapter 01***Back of Chapter Questions*****EXERCISE 1.1**

1. Determine whether each of the following relations are reflexive, symmetric and transitive:

(i) Relation R in the set $A = \{1, 2, 3, \dots, 13, 14\}$ defined as

$$R = \{(x, y) : 3x - y = 0\}$$

(ii) Relation R in the set N of natural numbers defined as

$$R = \{(x, y) : y = x + 5 \text{ and } x < 4\}$$

(iii) Relation R in the set $A = \{1, 2, 3, 4, 5, 6\}$ as

$$R = \{(x, y) : y \text{ is divisible by } x\}$$

(iv) Relation R in the set Z of all integers defined as

$$R = \{(x, y) : x - y \text{ is an integer}\}$$

(v) Relation R in the set A of human beings in a town at a particular time given by

(a) $R = \{(x, y) : x \text{ and } y \text{ work at the same place}\}$

(b) $R = \{(x, y) : x \text{ and } y \text{ live in the same locality}\}$

(c) $R = \{(x, y) : x \text{ is exactly 7 cm taller than } y\}$

(d) $R = \{(x, y) : x \text{ is wife of } y\}$

(e) $R = \{(x, y) : x \text{ is father of } y\}$

Solution:

(i) $A = \{1, 2, 3 \dots 13, 14\}$

$$R = \{(x, y) : 3x - y = 0\}$$

$$\text{Hence, } R = \{(1, 3), (2, 6), (3, 9), (4, 12)\}$$

R is not reflexive since $(1, 1), (2, 2) \dots (14, 14) \notin R$.

Again, R is not symmetric as $(1, 3) \in R$, but $(3, 1) \notin R$. [$3(3) - 1 \neq 0$]

Again, R is not transitive as $(1, 3), (3, 9) \in R$, but $(1, 9) \notin R$. [$3(1) - 9 \neq 0$]

Hence, R is neither reflexive, nor symmetric, nor transitive.

(ii) $A = \{1, 2, 3 \dots 13, 14\}$

$R = \{(x, y) : y = x + 5 \text{ and } x < 4\}$

Hence, $R = \{(1, 6), (2, 7), (3, 8)\}$

It is clear that $(1, 1) \notin R$.

$\therefore R$ is not reflexive.

$(1, 6) \in R$ But, $(6, 1) \notin R$.

$\therefore R$ is not symmetric.

Now, since there is no pair in R such that (x, y) and $(y, z) \in R$, so we will not bother about (x, z) .

$\therefore R$ is transitive.

Hence, R is neither reflexive, nor symmetric, but transitive.

(iii) $A = \{1, 2, 3, 4, 5, 6\}$

$R = \{(x, y) : y \text{ is divisible by } x\}$

We know that any number (x) is divisible by itself.

So, $(x, x) \in R$

$\therefore R$ is reflexive.

Now,

$(2, 4) \in R$ [as 4 is divisible by 2]

But, $(4, 2) \notin R$. [as 2 is not divisible by 4]

$\therefore R$ is not symmetric.

Suppose $(x, y), (y, z) \in R$. Then, y is divisible by x and z is divisible by y .

Hence, z is divisible by x .

$\Rightarrow (x, z) \in R$

∴ R is transitive.

Hence, R is reflexive and transitive but not symmetric.

(iv) $R = \{(x, y) : x - y \text{ is an integer}\}$

Now, for every $x \in Z$, $(x, x) \in R$ as $x - x = 0$ is an integer.

∴ R is reflexive.

Now, for every $y \in Z$, if $(x, y) \in R$, then $x - y$ is an integer.

⇒ $-(x - y)$ is also an integer.

⇒ $(y - x)$ is an integer.

∴ $(y, x) \in R$

∴ R is symmetric.

Now,

Suppose (x, y) and $(y, z) \in R$, where, $(y, z) \in Z$.

⇒ $(x - y)$ and $(y - z)$ are integers

⇒ $x - z = (x - y) + (y - z)$ is an integer.

∴ $(x, z) \in R$

∴ R is transitive.

Hence, R is reflexive, symmetric, and transitive.

(v)

(a) $R = \{(x, y) : x \text{ and } y \text{ work at the same place}\}$

⇒ $(x, x) \in R$ [as x and x work at the same place]

∴ R is reflexive.

If $(x, y) \in R$, then x and y work at the same place.

⇒ y and x work at the same place.

⇒ $(y, x) \in R$.

∴ R is symmetric.

Now, suppose $(x, y), (y, z) \in R$

⇒ x and y work at the same place and y and z work at the same place.

⇒ x and z work at the same place.

$$\Rightarrow (x, z) \in R$$

$\therefore R$ is transitive.

Hence, R is reflexive, symmetric and transitive.

$$(b) R = \{(x, y) : x \text{ and } y \text{ live in the same locality}\}$$

Clearly, $(x, x) \in R$ as x and x is the same human being.

$\therefore R$ is reflexive.

If $(x, y) \in R$, then x and y live in same locality.

$\Rightarrow y$ and x live in the same locality.

$$\Rightarrow (y, x) \in R$$

Hence, R is symmetric.

Now, suppose $(x, y) \in R$ and $(y, z) \in R$

$\Rightarrow x$ and y live in the same locality and y and z live in the same locality.

$\Rightarrow x$ and z live in the same locality.

$$\Rightarrow (x, z) \in R$$

R is transitive.

Hence, R is reflexive, symmetric and transitive.

$$(c) R = \{(x, y) : x \text{ is exactly } 7 \text{ cm taller than } y\}$$

Now, $(x, x) \notin R$

Since human being x cannot be taller than himself.

Hence, R is not reflexive.

Now, suppose $(x, y) \in R$.

$\Rightarrow x$ is exactly 7 cm taller than y .

Then, y is not taller than x . [Since, y is 7 cm smaller than x]

$$\therefore (y, x) \notin R$$

Indeed if x is exactly 7 cm taller than y , then y is exactly 7 cm shorter than x .

Hence, R is not symmetric.

Now,

Suppose $(x, y), (y, z) \in R$.

$\Rightarrow x$ is exactly 7 cm taller than y and y is exactly 7 cm taller than z .

$\Rightarrow x$ is exactly 14 cm taller than z .

$\therefore (x, z) \notin R$

$\therefore R$ is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

(d) $R = \{(x, y): x \text{ is the wife of } y\}$

Now,

$(x, x) \notin R$

Since x cannot be the wife of herself.

$\therefore R$ is not reflexive.

Now, suppose $(x, y) \in R$

$\Rightarrow x$ is the wife of y .

Clearly y is not the wife of x .

$\therefore (y, x) \notin R$

Indeed if x is the wife of y , then y is the husband of x .

R is not symmetric.

Suppose $(x, y), (y, z) \in R$

$\Rightarrow x$ is the wife of y and y is the wife of z .

This case is not possible. Also, this does not imply that x is the wife of z .

$\therefore (x, z) \notin R$

$\therefore R$ is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

(e) $R = \{(x, y): x \text{ is the father of } y\}$

$(x, x) \notin R$

As x cannot be the father of himself.

$\therefore R$ is not reflexive.

Now, suppose $(x, y) \in R$.

$\Rightarrow x$ is the father of y .

$\Rightarrow y$ cannot be the father of y .

Indeed, y is the son or the daughter of y .

$\therefore (y, x) \notin R$

$\therefore R$ is not symmetric.

Now, suppose $(x, y) \in R$ and $(y, z) \notin R$.

$\Rightarrow x$ is the father of y and y is the father of z .

$\Rightarrow x$ is not the father of z .

Indeed x is the grandfather of z .

$\therefore (x, z) \notin R$

$\therefore R$ is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

2. Show that the relation R in the set R of real numbers, defined as

$R = \{(a, b) : a \leq b^2\}$ is neither reflexive nor symmetric nor transitive.

Solution:

$R = \{(a, b) : a \leq b^2\}$

It can be observed that $(\frac{1}{2}, \frac{1}{2}) \notin R$, since, $\frac{1}{2} > (\frac{1}{2})^2$

$\therefore R$ is not reflexive.

Now, $(1, 4) \in R$ as $1 < 4^2$ But, 4 is not less than 1^2 .

$\therefore (4, 1) \notin R$

$\therefore R$ is not symmetric.

Now,

$(3, 2), (2, 1.5) \in R$ [as $3 < 2^2 = 4$ and $2 < (1.5)^2 = 2.25$]

But, $3 > (1.5)^2 = 2.25$

$\therefore (3, 1.5) \notin R$

$\therefore R$ is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

3. Check whether the relation R defined in the set $\{1, 2, 3, 4, 5, 6\}$ as

$R = \{(a, b) : b = a + 1\}$ is reflexive, symmetric or transitive.

Solution:

Suppose $A = \{1, 2, 3, 4, 5, 6\}$.

A relation R is defined on set A as: $R = \{(a, b) : b = a + 1\}$

$R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$

\Rightarrow we can find $(a, a) \notin R$, where $a \in A$.

For instance,

$(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6) \notin R$

$\therefore R$ is not reflexive.

\Rightarrow It can be observed that $(1, 2) \in R$, but $(2, 1) \notin R$.

Hence, R is not symmetric.

Now, $(1, 2), (2, 3) \in R$

But, $(1, 3) \notin R$

$\therefore R$ is not transitive

Hence, R is neither reflexive, nor symmetric, nor transitive.

4. Show that the relation R in R defined as $R = \{(a, b) : a \leq b\}$, is reflexive and

transitive but not symmetric.

Solution:

$$R = \{(a, b) : a \leq b\}$$

\Rightarrow Clearly $(a, a) \in R$ [as $a = a$]

R is reflexive.

\Rightarrow Now, $(2, 4) \in R$ (as $2 < 4$)

But, $(4, 2) \notin R$ as 4 is greater than 2.

$\therefore R$ is not symmetric.

Now, suppose $(a, b), (b, c) \in R$.

Then, $a \leq b$ and $b \leq c$

$\Rightarrow a \leq c$

$\Rightarrow (a, c) \in R$

$\therefore R$ is transitive.

Hence R is reflexive and transitive but not symmetric.

5. Check whether the relation R in R defined by $R = \{(a, b) : a \leq b^3\}$ is reflexive, Symmetric or transitive.

Solution:

$$R = \{(a, b) : a \leq b^3\}$$

\Rightarrow It is found that $(\frac{1}{2}, \frac{1}{2}) \notin R$, since, $\frac{1}{2} > (\frac{1}{2})^3$

$\therefore R$ is not reflexive.

\Rightarrow Now, $(1, 2) \in R$ (as $1 < 2^3 = 8$)

But, $(2, 1) \notin R$ (as $2^3 > 1$)

∴ R is not symmetric.

$$\Rightarrow \text{For, } \left(3, \frac{3}{2}\right) \left(\frac{3}{2}, \frac{6}{5}\right) \in R,$$

Since, $3 < \left(\frac{3}{2}\right)^3$ and $\frac{3}{2} < \left(\frac{6}{5}\right)^3$

But $\left(3, \frac{6}{5}\right) \notin R$ as $3 > \left(\frac{6}{5}\right)^3$

R is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

6. Show that the relation R in the set $\{1, 2, 3\}$ given by $R = \{(1, 2), (2, 1)\}$ is symmetric but neither reflexive nor transitive.

Solution:

Suppose $A = \{1, 2, 3\}$.

A relation R on A is defined as $R = \{(1, 2), (2, 1)\}$

⇒ It is clear that $(1, 1), (2, 2), (3, 3) \notin R$.

∴ R is not reflexive.

⇒ As $(1, 2) \in R$ and $(2, 1) \in R$,

Hence, R is symmetric.

⇒ Now, $(1, 2)$ and $(2, 1) \in R$

However, $(1, 1) \notin R$

∴ R is not transitive.

Hence, R is symmetric but neither reflexive nor transitive.

7. Show that the relation R in the set A of all the books in a library of a college,

given by $R = \{(x, y) : x \text{ and } y \text{ have same number of pages}\}$ is an equivalence relation.

Solution:

Given: Set A is the set of all books in the library of a college.

$R = \{(x, y) : x \text{ and } y \text{ have the same number of pages}\}$

\Rightarrow since $(x, x) \in R$ as x and x has the same number of pages.

Hence, R is reflexive

Suppose $(x, y) \in R \Rightarrow x$ and y have the same number of pages.

So, y and x have the same number of pages.

$(y, x) \in R$

$\therefore R$ is symmetric.

\Rightarrow Now, suppose $(x, y) \in R$ and $(y, z) \in R$.

$\Rightarrow x$ and y have the same number of pages and y and z have the same number of pages.

$\Rightarrow x$ and z have the same number of pages.

$\Rightarrow (x, z) \in R$

Hence, R is transitive.

As R is reflexive, symmetric and also transitive, R is an equivalence relation.

8. Show that the relation R in the set $A = \{1, 2, 3, 4, 5\}$ given by

$R = \{(a, b) : |a - b| \text{ is even}\}$, is an equivalence relation. Show that all the elements of $\{1, 3, 5\}$ are related to each other and all the elements of $\{2, 4\}$ are related to each other. But no element of $\{1, 3, 5\}$ is related to any element of $\{2, 4\}$.

Solution:

$A = \{1, 2, 3, 4, 5\}$ and $R = \{(a, b) : |a - b| \text{ is even}\}$

\Rightarrow It is clear that for any element $a \in A$, we have $|a - a| = 0$ (which is even).

$\therefore R$ is reflexive.

Suppose $(a, b) \in R$.

$\Rightarrow |a - b|$ is even

$\Rightarrow |-(a - b)| = |b - a|$ is also even

$\Rightarrow (b, a) \in R$

$\therefore R$ is symmetric.

Now, suppose $(a, b) \in R$ and $(b, c) \in R$.

$\Rightarrow |a - b|$ is even and $|b - c|$ is even

$\Rightarrow (a - b)$ is even and $(b - c)$ is even

$\Rightarrow (a - c) = (a - b) + (b - c)$ is even [Sum of two even integers is even]

$\Rightarrow |a - c|$ is even.

$\Rightarrow (a, c) \in R$

$\therefore R$ is transitive.

Since, the relation R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

Now, all elements of the set $\{1, 2, 3\}$ are related to each other as all the elements of this subset are odd. Thus, the modulus of the difference between any two odd elements will be even.

Similarly, all elements of the set $\{2, 4\}$ are related to each other as all the elements of this subset are even.

Also, no element of the subset $\{1, 3, 5\}$ can be related to any element of $\{2, 4\}$ as all elements of $\{1, 3, 5\}$ are odd and all elements of $\{2, 4\}$ are even. Thus, the modulus of the difference between even and odd numbers (from each of these two subsets) will not be even. [as $1-2, 1-4, 3-2, 3-4, 5-2$ and $5-4$ all are odd]

9. Show that each of the relation R in the set $A = \{x \in Z : 0 \leq x \leq 12\}$, given by

(i) $R = \{(a, b) : |a - b| \text{ is a multiple of } 4\}$

(ii) $R = \{(a, b) : a = b\}$

is an equivalence relation. Find the set of all elements related to 1 in each case.

Solution:

$$A = \{x \in Z : 0 \leq x \leq 12\} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

(i) $R = \{(a, b) : |a - b| \text{ is a multiple of } 4\}$

For any element, $a \in A$,

we have $(a, a) \in R$ as $|a - a| = 0$ is a multiple of 4.

R is reflexive.

Now, suppose $(a, b) \in R \Rightarrow |a - b|$ is a multiple of 4.

$$\Rightarrow |-(a - b)| = |b - a| \text{ is a multiple of } 4.$$

$$\Rightarrow (b, a) \in R$$

$\therefore R$ is symmetric.

Now, suppose $(a, b), (b, c) \in R$.

$$\Rightarrow |a - b| \text{ is a multiple of } 4 \text{ and } |b - c| \text{ is a multiple of } 4.$$

$$\Rightarrow (a - b) \text{ is a multiple of } 4 \text{ and } (b - c) \text{ is a multiple of } 4.$$

$$\Rightarrow (a - c) = (a - b) + (b - c) \text{ is a multiple of } 4.$$

$$\Rightarrow |a - c| \text{ is a multiple of } 4.$$

$$\Rightarrow (a, c) \in R$$

$\therefore R$ is transitive.

Since, the relation R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

The set of elements related to 1 is $\{1, 5, 9\}$ as

$$|1 - 1| = 0 \text{ is a multiple of } 4.$$

$$|5 - 1| = 4 \text{ is a multiple of } 4.$$

$|9 - 1| = 8$ is a multiple of 4.

Hence, $\{1, 5, 9\}$ is the set of elements related to 1.

(ii) $R = \{(a, b) : a = b\}$

For any element $a \in A$, we have $(a, a) \in R$, since $a = a$.

$\therefore R$ is reflexive.

Now, suppose $(a, b) \in R$.

Hence, $a = b$

then $b = a \Rightarrow (b, a) \in R$

$\therefore R$ is symmetric.

Now, suppose $(a, b) \in R$ and $(b, c) \in R$.

$\Rightarrow a = b$ and $b = c$

$\Rightarrow a = c$

$\Rightarrow (a, c) \in R$

$\therefore R$ is transitive.

Since, the relation R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

The elements in R that are related to 1 will be those elements from set A which are equal to 1.

Hence, the set of elements related to 1 is $\{1\}$.

10. Give an example of a relation. Which is

- (i) Symmetric but neither reflexive nor transitive.
- (ii) Transitive but neither reflexive nor symmetric.
- (iii) Reflexive and symmetric but not transitive.
- (iv) Reflexive and transitive but not symmetric.
- (v) Symmetric and transitive but not reflexive.

Solution:

(i) Suppose $A = \{5, 6, 7\}$.

Define a relation R on A as $R = \{(5, 6), (6, 7), (7, 5), (6, 5), (7, 6), (5, 7)\}$

Relation R is not reflexive as $(5, 5), (6, 6), (7, 7) \notin R$.

Now, as $(5, 6) \in R$ and also $(6, 5) \in R$

Hence, R is symmetric.

$\Rightarrow (5, 6), (6, 5) \in R$, but $(5, 5) \notin R$

$\therefore R$ is not transitive.

Hence, relation $R = \{(5, 6), (6, 7), (7, 5), (6, 5), (7, 6), (5, 7)\}$ on $A = \{5, 6, 7\}$ is the required relation.

(ii) Consider a relation R in R defined as:

$$R = \{(a, b) : a < b\}$$

For any $a \in R$, we have $(a, a) \notin R$ since a cannot be strictly less than a itself.

In fact, $a = a$.

$\therefore R$ is not reflexive.

Now, $(1, 2) \in R$ (as $1 < 2$)

But, 2 is not less than 1 .

$\therefore (2, 1) \notin R$

$\therefore R$ is not symmetric.

Now, suppose $(a, b), (b, c) \in R$.

$\Rightarrow a < b$ and $b < c$

$\Rightarrow a < c$

$\Rightarrow (a, c) \in R$

$\therefore R$ is transitive.

Hence, relation $R = \{(a, b) : a < b\}$ is transitive but neither reflexive nor symmetric.

(iii) Suppose $A = \{4, 6, 8\}$.

Define a relation R on A as

$$R = \{(4, 4), (6, 6), (8, 8), (4, 6), (6, 4), (6, 8), (8, 6)\}$$

Relation R is reflexive since for every $a \in A$, $(a, a) \in R$

i.e., $\{(4,4), (6,6), (8,8)\} \in R$.

Relation R is symmetric since $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in R$.

Relation R is not transitive

since $(4, 6), (6, 8) \in R$, but $(4, 8) \notin R$.

Hence, relation $R = \{(4,4), (6,6), (8,8), (4,6), (6,4), (6,8), (8,6)\}$ on $A = \{4, 6, 8\}$ is reflexive and symmetric but not transitive.

(iv) Define a relation R in R as:

$$R = \{a, b\}: a^3 \geq b^3\}$$

Clearly $(a, a) \in R$ as $a^3 = a^3$.

$\therefore R$ is reflexive.

Now, $(2, 1) \in R$ [as $2^3 \geq 1^3$]

But, $(1, 2) \notin R$ [as $1^3 < 2^3$]

$\therefore R$ is not symmetric.

Now, Suppose $(a, b), (b, c) \in R$.

$$\Rightarrow a^3 \geq b^3 \text{ and } b^3 \geq c^3$$

$$\Rightarrow a^3 \geq c^3$$

$$\Rightarrow (a, c) \in R$$

$\therefore R$ is transitive.

Hence, relation $R = \{a, b\}: a^3 \geq b^3\}$ in R is reflexive and transitive not symmetric.

(v) Suppose $A = \{0, 1, 2\}$.

Define a relation R on A as

$$R = \{(0,0), (1,1), (0,1), (1,0)\}$$

Relation R is not reflexive as $(2,2) \notin R$.

Relation R is symmetric as $(0,1) \in R$ and $(1,0) \in R$.

It is seen that $(0,1), (1,0) \in R$. Also, $(0,0) \in R$.

Also, while going through all possibilities, we can say that:

The relation R is transitive.

Hence, relation R is symmetric and transitive but not reflexive.

11. Show that the relation R in the set A of points in a plane given by $R = \{(P, Q) : \text{Distance of the point } P \text{ from the origin is same as the distance of the point } Q \text{ from the origin}\}$, is an equivalence relation. Further, show that the set of all points related to a point $P \neq (0, 0)$ is the circle passing through P with origin as centre.

Solution:

$R = \{(P, Q) : \text{Distance of point } P \text{ from the origin is the same as the distance of point } Q \text{ from the origin}\}$

Clearly, $(P, P) \in R$ since the distance of point P from the origin is always the same as the distance of the same point P from the origin.

$\therefore R$ is reflexive.

Now, Suppose $(P, Q) \in R$.

\Rightarrow The distance of point P from the origin is the same as the distance of point Q from the origin.

\Rightarrow The distance of point Q from the origin is the same as the distance of point P from the origin.

$\Rightarrow (Q, P) \in R$

$\therefore R$ is symmetric.

Now, Suppose $(P, Q), (Q, S) \in R$.

\Rightarrow The distance of points P and Q from the origin is the same and also, the distance of points Q and S from the origin is the same.

\Rightarrow The distance of points P and S from the origin is the same.

$\Rightarrow (P, S) \in R$

$\therefore R$ is transitive.

Since, the relation R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

The set of all points related to $P \neq (0, 0)$ will be those points whose distance from the origin is the same as the distance of point P from the origin.

In other words, if $O(0,0)$ is the origin and $OP = k$, then the set of all the points related to P is at a distance of k from the origin.

Hence, this set of points form a circle with the center as the origin and this circle passes through point P .

12. Show that the relation R defined in the set A of all triangles as $R = \{(T_1, T_2): T_1 \text{ is similar to } T_2\}$, is equivalence relation. Consider three right angle triangles T_1 with sides 3, 4, 5, T_2 with sides 5, 12, 13 and T_3 with sides 6, 8, 10. Which triangles among T_1, T_2 and T_3 are related?

Solution:

$$R = \{(T_1, T_2): T_1 \text{ is similar to } T_2\}$$

R is reflexive since every triangle is similar to itself.

Further,

If $(T_1, T_2) \in R$, then T_1 is similar to T_2 .

$\Rightarrow T_2$ is similar to T_1 .

$\Rightarrow (T_2, T_1) \in R$

$\therefore R$ is symmetric.

Now,

Suppose $(T_1, T_2), (T_2, T_3) \in R$.

$\Rightarrow T_1$ is similar to T_2 and T_2 is similar to T_3 .

$\Rightarrow T_1$ is similar to T_3 .

$\Rightarrow (T_1, T_3) \in R$

$\therefore R$ is transitive.

Since, the relation R is reflexive, symmetric and transitive.

Thus, R is an equivalence relation.

Now,

We can observe that

$$\frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \frac{1}{2}$$

∴ The corresponding sides of triangles T_1 and T_3 are in the same ratio.

Then, triangle T_1 is similar to triangle T_3 .

Hence, T_1 is related to T_3 .

- 13.** Show that the relation R defined in the set A of all polygons as $R = \{(P_1, P_2) : P_1 \text{ and } P_2 \text{ have same number of sides}\}$, is an equivalence relation. What is the set of all elements in A related to the right-angle triangle T with sides 3, 4 and 5?

Solution:

$$R = \{(P_1, P_2) : P_1 \text{ and } P_2 \text{ have same the number of sides}\}$$

Since $(P_1, P_1) \in R$, as the same polygon has the same number of sides with itself.

Hence, R is reflexive,

Suppose $(P_1, P_2) \in R$.

⇒ P_1 and P_2 have the same number of sides.

⇒ P_2 and P_1 have the same number of sides.

⇒ $(P_2, P_1) \in R$

∴ R is symmetric.

Now,

Suppose $(P_1, P_2), (P_2, P_3) \in R$.

⇒ P_1 and P_2 have the same number of sides.

Also, P_2 and P_3 have the same number of sides.

⇒ P_1 and P_3 have the same number of sides.

⇒ $(P_1, P_3) \in R$

∴ R is transitive.

Since, the relation R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

The elements in A related to the right-angled triangle (T) with sides 3, 4, and 5 are those polygons which have 3 sides (Since T is a polygon with 3 sides).

Hence, the set of all elements in A related to triangle T is the set of all triangles.

14. Let L be the set of all lines in XY plane and R be the relation in L defined as

$R = \{(L_1, L_2) : L_1 \text{ is parallel to } L_2\}$. Show that R is an equivalence relation. Find the set of all lines related to the line $y = 2x + 4$.

Solution:

$$R = \{(L_1, L_2) : L_1 \text{ is parallel to } L_2\}$$

R is reflexive as any line L_1 is parallel to itself i.e., $(L_1, L_1) \in R$.

Now, suppose $(L_1, L_2) \in R$.

$\Rightarrow L_1$ is parallel to $L_2 \Rightarrow L_2$ is parallel to L_1 .

$\Rightarrow (L_2, L_1) \in R$

$\therefore R$ is symmetric.

Now, suppose $(L_1, L_2), (L_2, L_3) \in R$.

$\Rightarrow L_1$ is parallel to L_2 . Also, L_2 is parallel to L_3 .

$\Rightarrow L_1$ is parallel to L_3 .

$\therefore R$ is transitive.

Since, the relation R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

The set of all lines related to the line $y = 2x + 4$ is the set of all the lines that are parallel to the line $y = 2x + 4$.

Slope of line $y = 2x + 4$ is $m = 2$

It is known that parallel lines have the same slopes.

The line parallel to the given line is of the form $y = 2x + c$, where $c \in R$.

Hence, the set of all lines related to the given line is $y = 2x + c$, where $c \in R$.

15. Let R be the relation in the set $\{1, 2, 3, 4\}$ given by

$R = \{(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)\}$. Choose the correct answer.

- (A) R is reflexive and symmetric but not transitive.
- (B) R is reflexive and transitive but not symmetric.
- (C) R is symmetric and transitive but not reflexive.
- (D) R is an equivalence relation.

Solution:

Given, $R = \{(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)\}$

It is seen that $(a, a) \in R$, for every

$a \in \{1, 2, 3, 4\}$.

$\therefore R$ is reflexive.

It is seen that $(1, 2) \in R$, but $(2, 1) \notin R$.

Hence, R is not symmetric.

Also, it is observed that $(a, b), (b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in \{1, 2, 3, 4\}$.

R is transitive.

Hence, R is reflexive and transitive but not symmetric.

The correct answer is B .

16. Let R be the relation in the set N given by $R = \{(a, b): a = b - 2, b > 6\}$. Choose

the correct answer.

(A) $(2, 4) \in R$

(B) $(3, 8) \in R$

(C) $(6, 8) \in R$

(D) $(8, 7) \in R$

Solution:

Given, $R = \{(a, b) : a = b - 2, b > 6\}$

Now,

Since $b > 6$,

As 4 is not greater than 6, $(2, 4) \notin R$

Also, as $3 \neq 8 - 2$,

$(3, 8) \notin R$

As $8 \neq 7 - 2$

$\therefore (8, 7) \notin R$

Now, consider $(6, 8)$.

We have $8 > 6$ and also, $6 = 8 - 2$.

$(6, 8) \in R$

The correct answer is C.

EXERCISE 1.2

1. Show that the function $f: R_* \rightarrow R_*$ defined by $f(x) = \frac{1}{x}$ is one-one and onto, where R_* is the set of all non-zero real numbers. Is the result true, if the domain R_* is replaced by N with co-domain being same as R_* ?

Solution:

It is given that $f: R_+ \rightarrow R_+$ is defined by $f(x) = \frac{1}{x}$

For one-one:

Suppose $x, y \in R_+$ such that $f(x) = f(y)$

$$\Rightarrow \frac{1}{x} = \frac{1}{y}$$

$$\Rightarrow x = y$$

$\therefore f$ is one-one.

For onto:

It is clear that for $y \in R_+$, there exists $x = \frac{1}{y} \in R_+$ [as $y \neq 0$]

such that, $f(x) = \frac{1}{\left(\frac{1}{y}\right)} = y$

$\therefore f$ is onto.

Thus, the given function f is one - one and onto.

Now, consider function $g: N \rightarrow R_+$ defined by $g(x) = \frac{1}{x}$

We have,

$$g(x_1) = g(x_2) \Rightarrow \frac{1}{x_1} = \frac{1}{x_2} \Rightarrow x_1 = x_2$$

$\therefore g$ is one - one.

Further, it is clear that g is not onto as for $1.2 \in R_+$ there does not exist any x in N such that, $g(x) = 1.2$

Hence, function g is one-one but not onto.

Hence, result is not same when domain is changed from R_+ to N .

2. Check the injectivity and surjectivity of the following functions:

(i) $f: N \rightarrow N$ given by $f(x) = x^2$

(ii) $f: Z \rightarrow Z$ given by $f(x) = x^2$

(iii) $f: R \rightarrow R$ given by $f(x) = x^2$

(iv) $f: N \rightarrow N$ given by $f(x) = x^3$

(v) $f: Z \rightarrow Z$ given by $f(x) = x^3$

Solution:

(i) $f: N \rightarrow N$ is given by $f(x) = x^2$

It is seen that for $x, y \in N$, $f(x) = f(y) \Rightarrow x^2 = y^2 \Rightarrow x = y$.

$\therefore f$ is injective.

Now, $2 \in N$. But, there does not exist any x in N such that $f(x) = x^2 = 2$.

$\therefore f$ is not surjective.

Hence, function f is injective but not surjective.

(ii) $f: Z \rightarrow Z$ is given by $f(x) = x^2$

It is seen that $f(-1) = f(1) = 1$, but $-1 \neq 1$.

$\therefore f$ is not injective.

Now, $-2 \in Z$. But, there does not exist any element $x \in Z$ such that

$f(x) = -2$ or $x^2 = -2$.

$\therefore f$ is not surjective.

Hence, function f is neither injective nor surjective.

(iii) $f: R \rightarrow R$ is given by $f(x) = x^2$

It is seen that $f(-1) = f(1)$

$= (-1)^2 = (1)^2$

but $-1 \neq 1$.

$\therefore f$ is not injective.

Now, $-2 \in R$. But, there does not exist any element $x \in R$ such that

$f(x) = -2$ or $x^2 = -2$.

$\therefore f$ is not surjective.

Hence, function f is neither injective nor surjective.

(iv) $f: N \rightarrow N$ given by $f(x) = x^3$

It is seen that for $x, y \in N, f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$.

$\therefore f$ is injective.

Now, $2 \in N$. But, there does not exist any element $x \in N$ such that

$$f(x) = 2 \text{ or } x^3 = 2.$$

$\therefore f$ is not surjective

Hence, function f is injective but not surjective.

(v) $f: Z \rightarrow Z$ is given by $f(x) = x^3$

It is seen that for $x, y \in Z, f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$.

$\therefore f$ is injective.

Now, $2 \in Z$. But, there does not exist any element $x \in Z$ such that

$$f(x) = 2 \text{ or } x^3 = 2.$$

$\therefore f$ is not surjective.

Hence, function f is injective but not surjective.

3. Prove that the Greatest Integer Function $f: R \rightarrow R$, given by $f(x) = [x]$, is neither one-one nor onto, where $[x]$ denotes the greatest integer less than or equal to x .

Solution:

$f: R \rightarrow R$ is given by, $f(x) = [x]$

It is seen that $f(1.2) = [1.2] = 1$, $f(1.9) = [1.9] = 1$.

$\therefore f(1.2) = f(1.9)$, but $1.2 \neq 1.9$.

$\therefore f$ is not one-one.

Now, consider $0.8 \in R$.

It is known that $f(x) = [x]$ is always an integer. Thus, there does not exist any element $x \in R$ such that $f(x) = 0.8$.

$\therefore f$ is not onto.

Hence, the greatest integer function is neither one- one nor onto.

4. Show that the Modulus Function $f: R \rightarrow R$, given by $f(x) = |x|$, is neither one- one nor onto, where $|x|$ is x , if x is positive or 0 and $|x|$ is $-x$, if x is negative.

Solution:

$$f: R \rightarrow R \text{ is given by } f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

It is clear that

$$\Rightarrow f(-1) = |-1| = 1$$

$$\Rightarrow f(1) = |1| = 1$$

$$\therefore f(-1) = f(1), \text{ but } -1 \neq 1.$$

$\therefore f$ is not one-one.

Now, consider $-1 \in R$.

It is known that $f(x) = |x|$ is always non - negative. Thus, there does not exist any element x in domain R such that $f(x) = |x| = -1$.

$\therefore f$ is not onto.

Hence, the modulus function is neither one - one nor onto.

5. Show that the Signum Function $f: R \rightarrow R$, given by

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

$\} \text{ is neither one-one nor onto.}$

Solution:

$$f: R \rightarrow R \text{ is given by } f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

It is seen that $f(1) = f(2) = 1$, but $1 \neq 2$.

$\therefore f$ is not one - one.

Now, as $f(x)$ takes only 3 values (1, 0, or -1) for the element -2 in co-domain R , there does not exist any x in domain R such that $f(x) = -2$.

$\therefore f$ is not onto.

Hence, the Signum function is neither one - one nor onto.

6. Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and let $f = \{(1, 4), (2, 5), (3, 6)\}$ be a function from A to B . Show that f is one-one.

Solution:

It is given that $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$.

$f: A \rightarrow B$ is defined as $f = \{(1, 4), (2, 5), (3, 6)\}$

$$\therefore f(1) = 4, f(2) = 5, f(3) = 6$$

It is seen that the images of distinct elements of A in f are distinct.

Hence, function f is one-one.

7. In each of the following cases, state whether the function is one-one, onto or bijective. Justify your answer.
- (i) $f: R \rightarrow R$ defined by $f(x) = 3 - 4x$
- (ii) $f: R \rightarrow R$ defined by $f(x) = 1 + x^2$

Solution:

(i) $f: R \rightarrow R$ is defined as $f(x) = 3 - 4x$.

Suppose $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$$\Rightarrow 3 - 4x_1 = 3 - 4x_2$$

$$\Rightarrow -4x_1 = -4x_2$$

$$\Rightarrow x_1 = x_2$$

$\therefore f$ is one-one.

For any real number (y) in R , there exists $\frac{3-y}{4}$ in R such that

$$f\left(\frac{3-y}{4}\right) = 3 - 4\left(\frac{3-y}{4}\right) = y$$

$\therefore f$ is onto.

As function is both one-one and onto, f is bijective.

(ii) $f: R \rightarrow R$ is defined as $f(x) = 1 + x^2$

Suppose $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$$\Rightarrow 1 + x_1^2 = 1 + x_2^2$$

$$\Rightarrow x_1^2 = x_2^2$$

$$\Rightarrow x_1 = \pm x_2$$

$f(x_1) = f(x_2)$ does not imply that $x_1 = x_2$

For example $f(1) = f(-1) = 2$

$\therefore f$ is not one-one.

Consider an element -2 in co-domain R .

It is seen that $f(x) = 1 + x^2$ is positive for all $x \in R$.

Thus, there does not exist any x in domain R such that $f(x) = -2$.

$\therefore f$ is not onto.

f is neither one - one nor onto hence, it is not bijective.

8. Let A and B be sets. Show that $f: A \times B \rightarrow B \times A$ such that $(a, b) = (b, a)$ is bijective function.

Solution:

$f: A \times B \rightarrow B \times A$ is defined as $f(a, b) = (b, a)$.

Suppose $(a_1, b_1), (a_2, b_2) \in A \times B$ such that $f(a_1, b_1) = f(a_2, b_2)$

$$\Rightarrow (b_1, a_1) = (b_2, a_2)$$

$$\Rightarrow b_1 = b_2 \text{ and } a_1 = a_2$$

$$\Rightarrow (a_1, b_1) = (a_2, b_2)$$

$\therefore f$ is one-one.

Now, suppose $(b, a) \in B \times A$ be any element.

Then, there exists $(a, b) \in A \times B$ such that $(a, b) = (b, a)$. [By definition of f]

$\therefore f$ is onto.

As the function is both one-one and onto, f is bijective.

9. Let: $N \rightarrow N$ be defined by $f(n) = \left\{ \begin{array}{l} \frac{n+1}{2}, \text{ if } n \text{ is odd} \\ \frac{n}{2}, \text{ if } n \text{ is even} \end{array} \right\}$

for all $n \in N$. State whether the function f is bijective. Justify your answer.

Solution:

$f: N \rightarrow N$ is defined as $f(n) = \left\{ \begin{array}{l} \frac{n+1}{2}, \text{ if } n \text{ is odd} \\ \frac{n}{2}, \text{ if } n \text{ is even} \end{array} \right\}$

for all $n \in N$

for $x=1, 2$ where 1 is odd and 2 is even number.

$$f(1) = \frac{1+1}{2} = 1 \text{ and } f(2) = \frac{2}{2} = 1 \text{ [By definition of } f(n)\text{]}$$

$$f(1) = f(2), \text{ where } 1 \neq 2$$

$\therefore f$ is not one - one.

Consider a natural number (n) in co - domain N .

Case I: n is odd

$\therefore n = 2r + 1$ for some $r \in N$. Then, there exists $4r + 1 \in N$ such that

$$f(4r + 1) = \frac{4r + 1 + 1}{2} = 2r + 1$$

Case II: n is even

$\therefore n = 2r$ for some $r \in N$. Then, there exists $4r \in N$ such that

$$f(4r) = \frac{4r}{2} = 2r.$$

$\therefore f$ is onto.

As the given function is not one-one hence, f is not a bijective function.

10. Let $A = R - \{3\}$ and $B = R - \{1\}$. Consider the function $f: A \rightarrow B$ defined by $f(x) = \left(\frac{x-2}{x-3}\right)$. Is f one-one and onto? Justify your answer.

Solution:

$A = R - \{3\}$, $B = R - \{1\}$ and $f: A \rightarrow B$ defined by $f(x) = \left(\frac{x-2}{x-3}\right)$

Suppose $x, y \in A$ such that $f(x) = f(y)$

$$\Rightarrow \frac{x-2}{x-3} = \frac{y-2}{y-3}$$

$$\Rightarrow (x-2)(y-3) = (y-2)(x-3)$$

$$\Rightarrow xy - 3x - 2y + 6 = xy - 2x - 3y + 6$$

$$\Rightarrow -3x - 2y = -2x - 3y \Rightarrow x = y$$

$\therefore f$ is one - one.

Suppose $y \in B = R - \{1\}$. Then, $y \neq 1$.

The function f is onto if there exists $x \in A$ such that $f(x) = y$.

Now, $f(x) = y$

$$\Rightarrow \frac{x-2}{x-3} = y$$

$$\Rightarrow x - 2 = xy - 3y \Rightarrow x(1 - y) = -3y + 2$$

$$\Rightarrow x = \frac{2-3y}{1-y} \in A [y \neq 1]$$

Thus, for any $y \in B$, there exists $\frac{2-3y}{1-y} \in A$ such that

$$f\left(\frac{2-3y}{1-y}\right) = \frac{\left(\frac{2-3y}{1-y}\right)-2}{\left(\frac{2-3y}{1-y}\right)-3} = \frac{2-3y-2+2y}{2-3y-3+3y} = \frac{-y}{-1} = y$$

$\therefore f$ is onto.

As the given function is both one-one and onto.

Hence, f is bijective.

11. Let: $R \rightarrow R$ be defined as $f(x) = x^4$. Choose the correct answer.

- (A) f is one-one onto
- (B) f is many-one onto
- (C) f is one-one but not onto
- (D) f is neither one-one nor onto.

Solution:

$f: R \rightarrow R$ is defined as $f(x) = x^4$.

Suppose $x, y \in R$ such that $(x) = f(y)$.

$$\Rightarrow x^4 = y^4$$

$$\Rightarrow x = \pm y$$

$\therefore f(x) = f(y)$ does not imply that $x = y$.

For example $f(1) = f(-1) = 1$

$\therefore f$ is not one - one.

Consider an element 2 in co - domain R . It is clear that there does not exist any x in domain R such that $f(x) = 2$.

$\therefore f$ is not onto.

Hence, function f is neither one - one nor onto.

The correct answer is D.

12. Let $f: R \rightarrow R$ be defined as $f(x) = 3x$. Choose the correct answer.

- (A) f is one-one and onto
- (B) f is many-one onto
- (C) f is one-one but not onto
- (D) f is neither one-one nor onto.

Solution:

$f: R \rightarrow R$ is defined as $f(x) = 3x$.

Suppose $x, y \in R$ such that $f(x) = f(y)$.

$$\Rightarrow 3x = 3y$$

$$\Rightarrow x = y$$

$\therefore f$ is one - one.

Also, for any real number (y) in co-domain R , there exists $\frac{y}{3}$ in R such that

$$f\left(\frac{y}{3}\right) = 3\left(\frac{y}{3}\right) = y$$

$\therefore f$ is onto.

Hence, function f is one - one and onto.

The correct answer is A.

EXERCISE 1.3

1. Let $f: \{1, 3, 4\} \rightarrow \{1, 2, 5\}$ and $g: \{1, 2, 5\} \rightarrow \{1, 3\}$ be given by
 $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(1, 3), (2, 3), (5, 1)\}$ Write down gof .

Solution:

The functions $f: \{1, 3, 4\} \rightarrow \{1, 2, 5\}$ and $g: \{1, 2, 5\} \rightarrow \{1, 3\}$ are defined as

$$f = \{(1, 2), (3, 5), (4, 1)\} \text{ and } g = \{(1, 3), (2, 3), (5, 1)\}$$

$$gof(1) = g[f(1)] = g(2) = 3 \text{ [as } f(1) = 2 \text{ and } g(2) = 3]$$

$$gof(3) = g[f(3)] = g(5) = 1 \text{ [as } f(3) = 5 \text{ and } g(5) = 1]$$

$$gof(4) = g[f(4)] = g(1) = 3 \text{ [as } f(4) = 1 \text{ and } g(1) = 3]$$

$$\text{Hence, } gof = \{(1, 3), (3, 1), (4, 3)\}$$

2. Let f, g and h be functions from R to R . Show that

$$(f + g)oh = foh + goh$$

$$(f \cdot g)oh = (foh) \cdot (goh)$$

Solution:

$$\Rightarrow \text{To prove: } (f + g)oh = foh + goh$$

$$\text{LHS} = [(f + g)oh](x)$$

$$= (f + g)[h(x)]$$

$$= f[h(x)] + g[h(x)]$$

$$= (foh)(x) + (goh)(x)$$

$$= \{(foh)(x) + (goh)\}(x) = \text{RHS}$$

$$\therefore \{(f + g)oh\}(x) = \{(foh)(x) + (goh)\}(x) \text{ for all } x \in R$$

$$\text{Hence, } (f + g)oh = foh + goh$$

To Prove: $(fg)oh = (foh).(goh)$

$$\text{LHS} = [(f.g)oh](x)$$

$$= (f.g)[h(x)]$$

$$= f[h(x)].g[h(x)]$$

$$= (foh)(x) . (goh)(x)$$

$$= \{(foh).(goh)\}(x) = \text{RHS}$$

$$\therefore [(f.g)oh](x) = \{(foh).(goh)\}(x) \text{ for all } x \in R$$

Hence, $(f.g)oh = (foh).(goh)$

3. Find gof and fog , if

(i) $f(x) = |x|$ and $g(x) = |5x - 2|$

(ii) $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$

Solution:

(i). $f(x) = |x|$ and $g(x) = |5x - 2|$

$$\therefore gof(x) = g(f(x)) = g(|x|) = |5|x| - 2|$$

$$fog(x) = f(g(x)) = f(|5x - 2|) = ||5x - 2|| = |5x - 2|$$

(ii). $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$

$$\therefore gof(x) = g(f(x)) = g(8x^3) = (8x^3)^{\frac{1}{3}} = 2x$$

$$fog(x) = f(g(x)) = f(x^{\frac{1}{3}}) = 8(x^{\frac{1}{3}})^3 = 8x$$

4. If $f(x) = \frac{(4x+3)}{(6x-4)}$, $x \neq \frac{2}{3}$, show that $f \circ f(x) = x$, for all $x \neq \frac{2}{3}$. What is the inverse of f ?

Solution:

It is given that $f(x) = \frac{(4x+3)}{(6x-4)}$, $x \neq \frac{2}{3}$

$$(f \circ f)(x) = f(f(x)) = f\left(\frac{4x+3}{6x-4}\right) = \frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4}$$

$$= \frac{16x+12+18x-12}{24x+18-24x+16} = \frac{34x}{34}$$

$$= x$$

$$\therefore f \circ f(x) = x, \text{ for all } x \neq \frac{2}{3}.$$

$$\Rightarrow f \circ f = I_x$$

Hence, the given function f is invertible and the inverse of f is f itself.

5. State with reason whether following functions have inverse

(i) $f: \{1, 2, 3, 4\} \rightarrow \{10\}$ with

$$f = \{(1, 10), (2, 10), (3, 10), (4, 10)\}$$

(ii) $g: \{5, 6, 7, 8\} \rightarrow \{1, 2, 3, 4\}$ with

$$g = \{(5, 4), (6, 3), (7, 4), (8, 2)\}$$

(iii) $h: \{2, 3, 4, 5\} \rightarrow \{7, 9, 11, 13\}$ with

$$h = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$$

Solution:

(i) $f: \{1, 2, 3, 4\} \rightarrow \{10\}$ defined as $f = \{(1, 10), (2, 10), (3, 10), (4, 10)\}$

From the given definition of f , we found that f is a many one function as

$$f(1) = f(2) = f(3) = f(4) = 10$$

$\therefore f$ is not one - one.

Hence, function f does not have an inverse.

(ii) $g: \{5, 6, 7, 8\} \rightarrow \{1, 2, 3, 4\}$ defined as

$$g = \{(5, 4), (6, 3), (7, 4), (8, 2)\}$$

From the given definition of g , it is seen that g is a many one function as

$$g(5) = g(7) = 4.$$

$\therefore g$ is not one - one.

Hence, function g does not have an inverse.

(iii) $h: \{2, 3, 4, 5\} \rightarrow \{7, 9, 11, 13\}$ defined as

$$h = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$$

It is seen that all distinct elements of the set $\{2, 3, 4, 5\}$ have distinct images under h .

\therefore function h is one - one.

Also, h is onto since for every element y of the set $\{7, 9, 11, 13\}$, there exists an element x in the set $\{2, 3, 4, 5\}$, such that $h(x) = y$.

Thus, h is a one - one and onto function.

Hence, h has an inverse.

6. Show that $f: [-1, 1] \rightarrow R$, given by $f(x) = \frac{x}{(x+2)}$ is one-one. Find the inverse of the function $f: [-1, 1] \rightarrow \text{Range } f$.

(Hint: For $y \in \text{Range } f$, $y = f(x) = \frac{x}{x+2}$, for some x in $[-1, 1]$, i.e., $x = \frac{2y}{(1-y)}$)

Solution:

$$f: [-1, 1] \rightarrow R \text{ is given as } f(x) = \frac{x}{(x+2)}$$

For one - one

$$\text{Suppose } f(x) = f(y)$$

$$\Rightarrow \frac{x}{x+2} = \frac{y}{y+2}$$

$$\Rightarrow xy + 2x = xy + 2y$$

$$\Rightarrow 2x = 2y$$

$$\Rightarrow x = y$$

f is a one - one function.

It's clear that the function $f: [-1, 1] \rightarrow \text{Range } f$ is onto.

$\therefore f: [-1, 1] \rightarrow \text{Range } f$ is one - one and onto and hence, the inverse of the function $f: [-1, 1] \rightarrow \text{Range } f$ exists.

Suppose $g: \text{Range } f \rightarrow [-1, 1]$ be the inverse of f .

Suppose y be an arbitrary element of range f .

Since $f: [-1, 1] \rightarrow \text{Range } f$ is onto, we have

$$y = f(x) \text{ for some } x \in [-1, 1]$$

$$\Rightarrow y = \frac{x}{x+2}$$

$$\Rightarrow xy + 2y = x$$

$$\Rightarrow x(1 - y) = 2y$$

$$\Rightarrow x = \frac{2y}{1-y}, y \neq 1$$

Now, let us define $g: \text{Range } f \rightarrow [-1, 1]$ as

$$g(y) = \frac{2y}{1-y}, y \neq 1$$

Now,

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{x}{x+2}\right) = \frac{2\left(\frac{x}{x+2}\right)}{1 - \left(\frac{x}{x+2}\right)} = \frac{2x}{x+2-x} = \frac{2x}{2} = x$$

and

$$(f \circ g)(y) = f(g(y)) = f\left(\frac{2y}{1-y}\right) = \frac{\frac{2y}{1-y}}{\frac{2y}{1-y} + 2} = \frac{2y}{2y+2-2y} = \frac{2y}{2} = y$$

$$\therefore g \circ f = x = I_{[-1,1]} \text{ and } f \circ g = y = I_{\text{Range } f}$$

$$f^{-1} = g$$

Hence, the inverse of f is, $f^{-1}(y) = \frac{2y}{1-y}, y \neq 1$

7. Consider $f: R \rightarrow R$ given by $f(x) = 4x + 3$. Show that f is invertible. Find the inverse of f .

Solution:

$f: R \rightarrow R$ is given by, $f(x) = 4x + 3$

For one -one

Suppose $f(x) = f(y)$

$$\Rightarrow 4x + 3 = 4y + 3$$

$$\Rightarrow 4x = 4y$$

$$\Rightarrow x = y$$

$\therefore f$ is a one - one function.

For onto

For $y \in R$, suppose $y = 4x + 3$.

$$\Rightarrow x = \frac{y-3}{4} \in R$$

Hence, for any $y \in R$, there exists $x = \frac{y-3}{4} \in R$, such that

$$f(x) = f\left(\frac{y-3}{4}\right) = 4\left(\frac{y-3}{4}\right) + 3 = y.$$

$\therefore f$ is onto.

Thus, f is one- one and onto and hence, f^{-1} exists.

Let us define $g: R \rightarrow R$ by $g(x) = \frac{y-3}{4}$

Now,

$$(g \circ f)(x) = g(f(x)) = g(4x + 3) = \frac{(4x+3)-3}{4} = \frac{4x}{4} = x$$

and

$$(f \circ g)(y) = f(g(y)) = f\left(\frac{y-3}{4}\right) = 4\left(\frac{y-3}{4}\right) + 3 = y - 3 + 3 = y$$

$$\therefore g \circ f = f \circ g = I_R$$

Hence, f is invertible and the inverse of f is given by $f^{-1}(y) = g(y) = \frac{y-3}{4}$.

8. Consider $f: R_+ \rightarrow [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of f given by $f^{-1}(y) = \sqrt{y-4}$, where R_+ is the set of all non-negative real numbers.

Solution:

$f: R_+ \rightarrow [4, \infty)$ is given as $f(x) = x^2 + 4$.

For one - one

Suppose $f(x) = f(y)$

$$\Rightarrow x^2 + 4 = y^2 + 4$$

$$\Rightarrow x^2 = y^2$$

$$\Rightarrow x = y \text{ [as } x = y \in R_+]$$

$\therefore f$ is a one - one function.

For onto

For $\in [4, \infty)$, suppose $y = x^2 + 4$

$$\Rightarrow x^2 = y - 4 \geq 0 \text{ [as } y \geq 4]$$

$$\Rightarrow x = \sqrt{y-4} \geq 0$$

Hence, for any $y \in [4, \infty)$, there exists $x = \sqrt{y-4} \in R_+$, such that

$$f(x) = f(\sqrt{y-4}) = (\sqrt{y-4})^2 + 4 = y - 4 + 4 = y$$

$\therefore f$ is onto.

As the given function f is one - one and onto and hence, f^{-1} exists.

Let us define $g: [4, \infty) \rightarrow R_+$ by $g(y) = \sqrt{y-4}$

Now,

$$(g \circ f)(x) = g(f(x)) = g(x^2 + 4) = \sqrt{(x^2 + 4) - 4} = \sqrt{x^2} = x$$

and

$$(f \circ g)(y) = f(g(y)) = f(\sqrt{y-4}) = (\sqrt{y-4})^2 + 4 = y - 4 + 4 = y$$

$$\therefore g \circ f = f \circ g = I_R$$

Hence, f is invertible and the inverse of f is given by $f^{-1}(y) = g(y) = \sqrt{y-4}$

9. Consider $f: R_+ \rightarrow [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(y) = \left(\frac{(\sqrt{y+6})-1}{3}\right)$.

Solution:

Given:

$f: R_+ \rightarrow [-5, \infty)$ is given as $f(x) = 9x^2 + 6x - 5$.

Suppose x_1, x_2 are two values of x for which $f(x_1) = f(x_2)$

$$\therefore 9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$$

$$\Rightarrow 9(x_1 + x_2)(x_1 - x_2) + 6(x_1 - x_2) = 0$$

$$\Rightarrow 9(x_1 - x_2)(9x_1 + 9x_2 + 6) = 0$$

$$\because x_1, x_2 \geq 0$$

$$\therefore 9x_1 + 9x_2 + 6 \neq 0$$

$$\therefore x_1 - x_2 = 0$$

$$\therefore x_1 = x_2 \text{ if } f(x_1) = f(x_2)$$

$\therefore f(x)$ is one-one function.

Again, $f(x) = 9x^2 + 6x - 5$

$$\Rightarrow f(x) = (3x + 1)^2 - 6 \quad \dots(i)$$

$$\because x \geq 0$$

$$\therefore 3x + 1 \geq 1$$

$$\therefore (3x + 1)^2 \geq 1$$

$$\therefore (3x + 1)^2 - 6 \geq -5$$

$$\therefore f(x) \geq -5$$

$$\therefore \text{Range of } f(x) = [-5, \infty)$$

$$\therefore \text{Range} = \text{Co-domain}$$

$\therefore f(x)$ is onto function.

$\therefore f(x)$ is one-one and onto both.

$\therefore f(x)$ is invertible

Again from (i)

$$y = f(x) = (3x + 1)^2 - 6$$

$$\Rightarrow (3x + 1)^2 = y + 6$$

$$\Rightarrow 3x + 1 = \sqrt{y + 6} \quad (\because 3x + 1 \geq 0)$$

$$\Rightarrow x = \frac{\sqrt{y + 6} - 1}{3}$$

$$\therefore f^{-1}(y) = \frac{\sqrt{y + 6} - 1}{3}$$

Hence, the inverse of f is given by

$$f^{-1}(y) = \left(\frac{(\sqrt{y+6})-1}{3} \right)$$

10. Let $f: X \rightarrow Y$ be an invertible function. Show that f has unique inverse.

(Hint: suppose g_1 and g_2 are two inverses of f . Then for all $y \in Y$,

$f \circ g_1(y) = I_Y(y) = f \circ g_2(y)$. Use one-one ness of f).

Solution:

Suppose $f: X \rightarrow Y$ be an invertible function.

Also, suppose f has two inverses (say g_1 and g_2)

Then, for all $y \in Y$, we have

$$f \circ g_1(y) = I_Y(y) = f \circ g_2(y)$$

$$\Rightarrow f(g_1(y)) = f(g_2(y))$$

$$\Rightarrow g_1(y) = g_2(y) \quad [\text{as } f \text{ is invertible} \Rightarrow f \text{ is one-one}]$$

$$\Rightarrow g_1 = g_2$$

Hence, the given function, f has a unique inverse.

11. Consider $f: \{1, 2, 3\} \rightarrow \{a, b, c\}$ given by $f(1) = a, f(2) = b$ and $f(3) = c$. Find f^{-1} and show that $(f^{-1})^{-1} = f$.

Solution:

Function $f: \{1, 2, 3\} \rightarrow \{a, b, c\}$ is given by $f(1) = a, f(2) = b$, and $f(3) = c$

If we define $g: \{a, b, c\} \rightarrow \{1, 2, 3\}$ as $g(a) = 1, g(b) = 2, g(c) = 3$.

We have

$$(f \circ g)(a) = f(g(a)) = f(1) = a$$

$$(f \circ g)(b) = f(g(b)) = f(2) = b$$

$$(f \circ g)(c) = f(g(c)) = f(3) = c$$

and

$$(g \circ f)(1) = g(f(1)) = g(a) = 1$$

$$(g \circ f)(2) = g(f(2)) = g(b) = 2$$

$$(g \circ f)(3) = g(f(3)) = g(c) = 3$$

$\therefore g \circ f = I_X$ and $f \circ g = I_Y$, where $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$.

Thus, the inverse of f exists and $f^{-1} = g$.

$\therefore f^{-1}: \{a, b, c\} \rightarrow \{1, 2, 3\}$ is given by

$$f^{-1}(a) = 1$$

$$f^{-1}(b) = 2$$

$$f^{-1}(c) = 3$$

Let us now find the inverse of f^{-1} i.e., find the inverse of g .

If we define $h: \{1, 2, 3\} \rightarrow \{a, b, c\}$ as $h(1) = a, h(2) = b, h(3) = c$

We have

$$(g \circ h)(1) = g(h(1)) = g(a) = 1$$

$$(goh)(2) = g(h(2)) = g(b) = 2$$

$$(goh)(3) = g(h(3)) = g(c) = 3$$

and

$$(hog)(a) = h(g(a)) = h(1) = a$$

$$(hog)(b) = h(g(b)) = h(2) = b$$

$$(hog)(c) = h(g(c)) = h(3) = c$$

$\therefore goh = I_X$ and $hog = I_Y$, where $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$.

Thus, the inverse of g exists and $g^{-1} = h \Rightarrow (f^{-1})^{-1} = h$.

It can be noted that $h = f$

Hence, $(f^{-1})^{-1} = f$

12. Let $f: X \rightarrow Y$ be an invertible function. Show that the inverse of f^{-1} is f , i.e.,
 $(f^{-1})^{-1} = f$.

Solution:

Suppose $f: X \rightarrow Y$ be an invertible function.

Then, there exists a function $g: Y \rightarrow X$ such that $gof = I_X$ and $fog = I_Y$.

Here, $f^{-1} = g$.

Now, $gof = I_X$ and $fog = I_Y$

$\Rightarrow f^{-1}of = I_X$ and $fof^{-1} = I_Y$

Hence, $f^{-1}: Y \rightarrow X$ is invertible and f is the inverse of f^{-1} i.e., $(f^{-1})^{-1} = f$

13. If $f: R \rightarrow R$ be given by $f(x) = (3 - x^3)^{\frac{1}{3}}$, then $f \circ f(x)$ is

- (A) $x^{\frac{1}{3}}$
 (B) x^3
 (C) x
 (D) $(3 - x^3)$

Solution:

$f: R \rightarrow R$ be given as $f(x) = (3 - x^3)^{\frac{1}{3}}$

$$\therefore f \circ f(x) = f(f(x)) = f\left((3 - x^3)^{\frac{1}{3}}\right) = \left[3 - \left((3 - x^3)^{\frac{1}{3}}\right)^3\right]^{\frac{1}{3}}$$

$$= [3 - (3 - x^3)]^{\frac{1}{3}} = (x^3)^{\frac{1}{3}} = x$$

$$\therefore f \circ f(x) = x$$

The correct Answer is C.

14. Let $f: R - \left\{-\frac{4}{3}\right\} \rightarrow R$ be a function defined as $f(x) = \frac{4x}{3x+4}$. The inverse of f is the

map $g: \text{Range } f \rightarrow R - \left\{-\frac{4}{3}\right\}$ given by

(A) $g(y) = \frac{3y}{3-4y}$

(B) $g(y) = \frac{4y}{4-3y}$

(C) $g(y) = \frac{4y}{3-4y}$

(D) $g(y) = \frac{3y}{4-3y}$

Solution:

It is given that $f: R - \left\{-\frac{4}{3}\right\} \rightarrow R$ be a function as $f(x) = \frac{4x}{3x+4}$

Suppose y be an arbitrary element of Range f

Then, there exists $x \in R - \left\{-\frac{4}{3}\right\}$ such that $y = f(x)$

$$\Rightarrow y = \frac{4x}{3x+4}$$

$$\Rightarrow 3xy + 4y = 4x$$

$$\Rightarrow x(4 - 3y) = 4y$$

$$\Rightarrow x = \frac{4y}{4-3y}$$

Let us define $g: \text{Range } f \rightarrow R - \left\{-\frac{4}{3}\right\}$ as $g(y) = \frac{4y}{4-3y}$

Now,

$$g \circ f(x) = g(f(x)) = g\left(\frac{4x}{3x+4}\right) = \frac{4\left(\frac{4x}{3x+4}\right)}{4-3\left(\frac{4x}{3x+4}\right)}$$

$$= \frac{16x}{12x+16-12x} = \frac{16x}{16} = x$$

and

$$f \circ g(y) = f(g(y)) = f\left(\frac{4y}{4-3y}\right) = \frac{4\left(\frac{4y}{4-3y}\right)}{3\left(\frac{4y}{4-3y}\right)+4}$$

$$= \frac{16y}{12y+16-12y} = \frac{16y}{16} = y$$

$$\therefore g \circ f = I_{R - \left\{-\frac{4}{3}\right\}} \text{ and } f \circ g = I_{\text{Range } f}$$

Thus, g is the inverse of f i.e., $f^{-1} = g$.

Hence, the inverse of f is the map $g: \text{Range } f \rightarrow R - \left\{-\frac{4}{3}\right\}$, which is given by

$$g(y) = \frac{4y}{4-3y}.$$

The correct Answer is B.

EXERCISE 1.4

1. Determine whether or not each of the definition of $*$ given below gives a binary operation. In the event that $*$ is not a binary operation, give justification for this.
- (i) On Z^+ , define $*$ by $a * b = a - b$
 - (ii) On Z^+ , define $*$ by $a * b = ab$
 - (iii) On R , define $*$ by $a * b = ab^2$
 - (iv) On Z^+ , define $*$ by $a * b = |a - b|$
 - (v) On Z^+ , define $*$ by $a * b = a$

Solution:

(i) On Z^+ , $*$ is defined by $a * b = a - b$.

Here, the image of $(1, 2)$ under $*$ is $1 * 2 = 1 - 2 = -1 \notin Z^+$.

Hence, the given definition of $*$ is not a binary operation.

(ii) On Z^+ , $*$ is defined by $a * b = ab$.

It is clear that for each $a, b \in Z^+$, there is a unique element ab in Z^+ .

This means that $*$ takes each pair (a, b) to a unique element $a * b = ab$ in Z^+ .

Hence, $*$ is a binary operation.

(iii) On R , $*$ is defined by $a * b = ab^2$.

It is clear that for each $a, b \in R$, there is a unique element ab^2 in R .

This means that $*$ takes each pair (a, b) to a unique element $a * b = ab^2$ in R .

Hence, $*$ is a binary operation.

(iv) On Z^+ , $*$ is defined by $a * b = |a - b|$.

Here, the image of $(1, 1)$ under $*$ is $1 * 1 = |1 - 1| = 0 \notin Z^+$

Hence, $*$ is not a binary operation.

(v) On Z^+ , $*$ is defined by $a * b = a$.

It is clear that for each $a, b \in Z^+$, there is a unique element a in Z^+ .

This means that $*$ takes each pair (a, b) to a unique element $a * b = a$ in Z^+ .

Hence, $*$ is a binary operation.

2. For each operation $*$ defined below, determine whether $*$ is binary, commutative or associative.

(i) On Z , define $a * b = a - b$

(ii) On Q , define $a * b = ab + 1$

(iii) On Q , define $a * b = \frac{ab}{2}$

(iv) On Z^+ , define $a * b = 2^{ab}$

(v) On Z^+ , define $a * b = a^b$

(vi) On $R - \{-1\}$, define $a * b = \frac{a}{b+1}$

Solution:

(i) On Z , $*$ is defined by $a * b = a - b$.

If $a, b \in Z$, then $a - b \in Z$.

Hence, the operation $*$ is a binary operation.

It is observed that $1 * 2 = 1 - 2 = -1$ and $2 * 1 = 2 - 1 = 1$.

$\therefore 1 * 2 \neq 2 * 1$, where $1, 2 \in Z$

Hence, the operation $*$ is not commutative.

Also, we have

$$(1 * 2) * 3 = (1 - 2) * 3 = -1 * 3 = -1 - 3 = -4$$

$$1 * (2 * 3) = 1 * (2 - 3) = 1 * -1 = 1 - (-1) = 2$$

$\therefore (1 * 2) * 3 \neq 1 * (2 * 3)$, where $1, 2, 3 \in Z$

Hence, the operation $*$ is not associative.

(ii) On Q , $*$ is defined by $a * b = ab + 1$.

If $a, b \in Q$, then $ab + 1 \in Q$.

Hence, the operation $*$ is a binary operation.

We know that: $ab = ba$ for all $a, b \in Q$

$$\Rightarrow ab + 1 = ba + 1 \text{ for all } a, b \in Q$$

$$\Rightarrow a * b = b * a \text{ for all } a, b \in Q$$

Hence, the operation $*$ is commutative.

It can be observed that

$$(1 * 2) * 3 = (1 \times 2 + 1) * 3 = 3 * 3 = 3 \times 3 + 1 = 10$$

$$1 * (2 * 3) = 1 * (2 \times 3 + 1) = 1 * 7 = 1 \times 7 + 1 = 8$$

$$(1 * 2) * 3 \neq 1 * (2 * 3), \text{ where } 1, 2, 3 \in Q$$

Hence, the operation $*$ is not associative.

Hence, the operation $*$ is commutative and not associative.

(iii) On Q , $*$ is defined by $a * b = \frac{ab}{2}$

If $a, b \in Q$, then $\frac{ab}{2} \in Q$.

Hence, the operation $*$ is a binary operation.

We know that: $ab = ba$ for all $a, b \in Q$

$$\Rightarrow \frac{ab}{2} = \frac{ba}{2} \text{ for all } a, b \in Q$$

$$\Rightarrow a * b = b * a \text{ for all } a, b \in Q$$

Hence, the operation $*$ is commutative.

Again, for all $a, b, c \in Q$, we have

$$(a * b) * c = \left(\frac{ab}{2}\right) * c = \frac{\left(\frac{ab}{2}\right)c}{2} = \frac{abc}{4}$$

and

$$a * (b * c) = a * \left(\frac{bc}{2}\right) = \frac{a\left(\frac{bc}{2}\right)}{2} = \frac{abc}{4}$$

$$\therefore (a * b) * c = a * (b * c), \text{ where } a, b, c \in Q$$

Hence, the operation $*$ is associative.

Hence, the operation $*$ is associative and commutative.

(iv) On Z^+ , $*$ is defined by $a * b = 2^{ab}$.

If $a, b \in Z^+$, then $2^{ab} \in Z^+$.

Hence, the operation $*$ is a binary operation.

we know that: $ab = ba$ for all $a, b \in Z^+$

$$\Rightarrow 2^{ab} = 2^{ba} \text{ for all } a, b \in Z^+$$

$$\Rightarrow a * b = b * a \text{ for all } a, b \in Z^+$$

Hence, the operation $*$ is commutative.

It is observed that

$$(1 * 2) * 3 = 2^{1 \times 2} * 3 = 4 * 3 = 2^{4 \times 3} = 2^{12} \text{ and}$$

$$1 * (2 * 3) = 1 * 2^{2 \times 3} = 1 * 2^6 = 1 * 64 = 2^{1 \times 64} = 2^{64}$$

Since, $(1 * 2) * 3 \neq 1 * (2 * 3)$, where $1, 2, 3 \in Z^+$

Hence, the operation $*$ is not associative.

Hence, the operation $*$ is commutative and not associative.

(v) On Z^+ , $*$ is defined by $a * b = a^b$.

If $a, b \in Z^+$, then $a^b \in Z^+$.

Hence, the operation $*$ is a binary operation.

It is observed that

$$1 * 2 = 1^2 = 1 \text{ and } 2 * 1 = 2^1 = 2$$

Since, $1 * 2 \neq 2 * 1$, where $1, 2 \in Z^+$

Hence, the operation $*$ is not commutative.

It is also observed that

$$(2 * 3) * 4 = 2^3 * 4 = 8 * 4 = 8^4 = 2^{12} \text{ and}$$

$$2 * (3 * 4) = 2 * 3^4 = 2 * 81 = 2^{81}$$

$(2 * 3) * 4 \neq 2 * (3 * 4)$, where $2, 3, 4 \in Z^+$

Hence, the operation $*$ is not associative.

Hence, the operation $*$ is neither associative nor commutative.

(vi) On $R - \{-1\}$, $*$ is defined by $a * b = \frac{a}{b+1}$

If $a, b \in R - \{-1\}$, then $\frac{a}{b+1} \in R - \{-1\}$.

Hence, the operation $*$ is a binary operation.

It is observed that

$$1 * 2 = \frac{1}{2+1} = \frac{1}{3} \text{ and } 2 * 1 = \frac{2}{1+1} = \frac{2}{2} = 1$$

Since, $1 * 2 \neq 2 * 1$, where $1, 2 \in R - \{-1\}$

Hence, the operation $*$ is not commutative.

It is also observed that

$$(1 * 2) * 3 = \frac{1}{2+1} * 3 = \frac{1}{3} * 3 = \frac{1}{3+1} = \frac{1}{4}$$

and

$$1 * (2 * 3) = 1 * \frac{2}{3+1} = 1 * \frac{2}{4} = 1 * \frac{1}{2} = \frac{1}{\frac{1}{2}+1} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

Since, $(1 * 2) * 3 \neq 1 * (2 * 3)$, where $1, 2, 3 \in R - \{-1\}$

Hence, the operation $*$ is not associative.

Hence, the operation $*$ is neither associative nor commutative.

3. Consider the binary operation \wedge on the set $\{1, 2, 3, 4, 5\}$ defined by $a \wedge b = \min \{a, b\}$. Write the operation table of the operation \wedge .

Solution:

Since, the binary operation on the set $\{1, 2, 3, 4, 5\}$ is defined as $a \wedge b = \min \{a, b\}$ for all $a, b \in \{1, 2, 3, 4, 5\}$.

Hence, the operation table for the given operation \wedge can be given as:

\wedge	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

4. Consider a binary operation $*$ on the set $\{1, 2, 3, 4, 5\}$ given by the following multiplication table.

(i) Compute $(2 * 3) * 4$ and $2 * (3 * 4)$

(ii) Is $*$ commutative?

(iii) Compute $(2 * 3) * (4 * 5)$.

(Hint: use the following table)

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

Solution:

(i) We need to find: $(2 * 3) * 4$ and $2 * (3 * 4)$

Using table,

$$(2 * 3) * 4 = 1 * 4 = 1$$

$$2 * (3 * 4) = 2 * 1 = 1$$

(ii) For every $a, b \in \{1, 2, 3, 4, 5\}$, we can observe that $a * b = b * a$. Hence, the operation

$*$ is commutative.

(iii) From table, $(2 * 3) = 1$ and $(4 * 5) = 1$

$$\therefore (2 * 3) * (4 * 5) = 1 * 1 = 1$$

5. Let $*'$ be the binary operation on the set $\{1, 2, 3, 4, 5\}$ defined by $a *' b = H.C.F.$ of a and b . Is the operation $*'$ same as the operation $*$ defined in question 4 above? Justify your answer.

Solution:

Given, the binary operation $*$ ' on the set $\{1, 2, 3, 4, 5\}$ is defined as $a *' b = H.C.F$ of a and b .

Hence, the operation table for the operation $*$ ' is given as:

$*$ '	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

We can see that the operation tables for the operations $*$ and $*$ ' are the same.

Hence, the operation $*$ ' is same as the operation $*$.

6. Let $*$ be the binary operation on N given by $a * b = L.C.M.$ of a and b . Find
- $5 * 7, 20 * 16$
 - Is $*$ commutative?
 - Is $*$ associative?
 - Find the identity of $*$ in N
 - Which elements of N are invertible for the operation $*$?

Solution:

Since, the binary operation $*$ on N is defined as $a * b = L.C.M.$ of a and b .

(i) Hence, $5 * 7 = L.C.M.$ of 5 and 7 = 35

$20 * 16 = L.C.M$ of 20 and 16 = 80

(ii) As we know that

$L.C.M$ of a and $b = L.C.M$ of b and a for all $a, b \in N$.

Hence, $a * b = b * a$

Hence, the operation $*$ is commutative.

(iii) For $a, b, c \in N$, we have

$(a * b) * c = (L.C.M$ of a and $b) * c = LCM$ of a, b , and c

$a * (b * c) = a * (LCM$ of b and $c) = L.C.M$ of a, b , and c

$\therefore (a * b) * c = a * (b * c)$

Hence, the operation $*$ is associative.

(iv) As we know that,

$L.C.M.$ of a and $1 = a = L.C.M.$ 1 and a for all $a \in N$

$\Rightarrow a * 1 = a = 1 * a$ for all $a \in N$

Hence, 1 is the identity of $*$ in N .

(v) An element a in N is invertible with respect to the operation $*$ if and only if there exists

an element b in N , such that $a * b = e = b * a$.

Here, $e = 1$

It means that

$L.C.M$ of a and $b = 1 = L.C.M$ of b and a

This is possible only if a and b are equal to 1 .

Hence, 1 is the only invertible element of N with respect to the operation $*$.

7. Is $*$ defined on the set $\{1, 2, 3, 4, 5\}$ by $a * b = L.C.M.$ of a and b a binary operation? Justify your answer.

Solution:

Given, the operation $*$ on the set $A = \{1, 2, 3, 4, 5\}$ as $a * b = L.C.M.$ of a

and b .

Hence, the operation table for the given operation $*$ is:

$*$	1	2	3	4	5
1	1	2	3	4	5
2	2	2	6	4	10
3	3	6	3	12	15
4	4	4	12	4	20
5	5	10	15	20	5

From the obtained table, we can observe that,

$$3 * 2 = 2 * 3 = 6 \notin A,$$

$$5 * 2 = 2 * 5 = 10 \notin A,$$

$$3 * 4 = 4 * 3 = 12 \notin A,$$

$$3 * 5 = 5 * 3 = 15 \notin A,$$

$$4 * 5 = 5 * 4 = 20 \notin A$$

Hence, the given operation $*$ is not a binary operation.

8. Let $*$ be the binary operation on N defined by $a * b = H.C.F.$ of a and b . Is $*$ commutative? Is $*$ associative? Does there exist identity for this binary operation on N ?

Solution:

Given, the binary operation $*$ on N as: $a * b = H.C.F.$ of a and b

As we know that,

$H.C.F.$ of a and $b = H.C.F.$ of b and a for all $a, b \in N$.

$$\therefore a * b = b * a$$

Hence, the operation $*$ is commutative.

For $a, b, c \in N$, we have

$$(a * b) * c = (H.C.F. \text{ of } a \text{ and } b) * c = H.C.F. \text{ of } a, b \text{ and } c$$

$$a * (b * c) = a * (H.C.F. \text{ of } b \text{ and } c) = H.C.F. \text{ of } a, b, \text{ and } c$$

$$\therefore (a * b) * c = a * (b * c)$$

Hence, the operation $*$ is associative.

Now, an element $e \in N$ will be the identity for the operation $*$ if $a * e = a = e * a$ for all $a \in N$.

But this is not true for any $a \in N$.

Hence, the operation $*$ does not have any identity in N .

9. Let $*$ be a binary operation on the set Q of rational numbers as follows:

(i) $a * b = a - b$

(ii) $a * b = a^2 + b^2$

(iii) $a * b = a + ab$

(iv) $a * b = (a - b)^2$

(v) $a * b = \frac{ab}{4}$

(vi) $a * b = ab^2$

Find which of the binary operations are commutative and which are associative.

Solution:

(i) On Q , the binary operation $*$ is defined as $a * b = a - b$. It is observed that:

$$\frac{1}{2} * \frac{1}{3} = \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}$$

and

$$\frac{1}{3} * \frac{1}{2} = \frac{1}{3} - \frac{1}{2} = \frac{2-3}{6} = \frac{-1}{6}$$

$$\therefore \frac{1}{2} * \frac{1}{3} \neq \frac{1}{3} * \frac{1}{2}, \text{ where } \frac{1}{2}, \frac{1}{3} \in Q$$

Hence, the operation $*$ is not commutative.

It is also observed that

$$\left(\frac{1}{2} * \frac{1}{3}\right) * \frac{1}{4} = \left(\frac{1}{2} - \frac{1}{3}\right) * \frac{1}{4} = \left(\frac{3-2}{6}\right) * \frac{1}{4} = \frac{1}{6} * \frac{1}{4} = \frac{1}{6} - \frac{1}{4} = \frac{2-3}{12} = \frac{-1}{12}$$

and

$$\frac{1}{2} * \left(\frac{1}{3} * \frac{1}{4}\right) = \frac{1}{2} * \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{2} * \left(\frac{4-3}{12}\right) = \frac{1}{2} * \frac{1}{12} = \frac{1}{2} - \frac{1}{12} = \frac{6-1}{12} = \frac{5}{12}$$

Since, $\left(\frac{1}{2} * \frac{1}{3}\right) * \frac{1}{4} \neq \frac{1}{2} * \left(\frac{1}{3} * \frac{1}{4}\right)$, where $\frac{1}{2}, \frac{1}{3}, \frac{1}{4} \in Q$

Hence, the operation $*$ is not associative.

(ii) On Q , the binary operation $*$ is defined as $a * b = a^2 + b^2$.

For $a, b \in Q$, we have

$$a * b = a^2 + b^2 = b^2 + a^2 = b * a$$

$$\therefore a * b = b * a$$

Hence, the operation $*$ is commutative.

It is also found that,

$$(1 * 2) * 3 = (1^2 + 2^2) * 3 = (1 + 4) * 3 = 5 * 3 = 5^2 + 3^2 = 34 \text{ and}$$

$$1 * (2 * 3) = 1 * (2^2 + 3^2) = 1 * (4 + 9) = 1 * 13 = 1^2 + 13^2 = 170$$

$$\therefore (1 * 2) * 3 \neq 1 * (2 * 3), \text{ where } 1, 2, 3 \in Q$$

Hence, the operation $*$ is not associative.

(iii) On Q , the binary operation $*$ is defined as $a * b = a + ab$.

It is found that

$$1 * 2 = 1 + 1 \times 2 = 1 + 2 = 3$$

$$2 * 1 = 2 + 2 \times 1 = 2 + 2 = 4$$

Since, $1 * 2 \neq 2 * 1$, where $1, 2 \in Q$

Hence, the operation $*$ is not commutative.

It is also observed that

$$(1 * 2) * 3 = (1 + 1 \times 2) * 3 = (1 + 2) * 3 = 3 * 3 = 3 + 3 \times 3 = 3 + 9 = 12 \text{ and}$$

$$\therefore 1 * (2 * 3) = 1 * (2 + 2 \times 3) = 1 * (2 + 6) = 1 * 8 = 1 + 1 \times 8 = 1 + 8 = 9$$

$$\therefore (1 * 2) * 3 \neq 1 * (2 * 3), \text{ where } 1, 2, 3 \in Q$$

Hence, the operation $*$ is not associative.

(iv) On Q , the binary operation $*$ is defined by $a * b = (a - b)^2$.

For $a, b \in Q$, we have

$$a * b = (a - b)^2$$

$$b * a = (b - a)^2 = [-(a - b)]^2 = (a - b)^2$$

$$\text{Since, } a * b = b * a$$

Hence, the operation $*$ is commutative.

Now, it is also seen that

$$(1 * 2) * 3 = (1 - 2)^2 * 3 = (-1)^2 * 3 = 1 * 3 = (1 - 3)^2 = (-2)^2 = 4$$

and

$$1 * (2 * 3) = 1 * (2 - 3)^2 = 1 * (-1)^2 = 1 * 1 = (1 - 1)^2 = 0$$

Since, $(1 * 2) * 3 \neq 1 * (2 * 3)$, where $1, 2, 3 \in Q$

Hence, the operation $*$ is not associative.

(v) On Q , the binary operation $*$ is defined as $a * b = \frac{ab}{4}$.

For $a, b \in Q$, we have

$$a * b = \frac{ab}{4} = \frac{ba}{4} = b * a$$

$$a * b = b * a$$

Hence, the operation $*$ is commutative.

For $a, b, c \in Q$, we have

$$(a * b) * c = \left(\frac{ab}{4}\right) * c = \frac{\left(\frac{ab}{4}\right) \cdot c}{4} = \frac{abc}{16}$$

and

$$a * (b * c) = a * \left(\frac{bc}{4}\right) = \frac{a \cdot \left(\frac{bc}{4}\right)}{4} = \frac{abc}{16}$$

$\therefore (a * b) * c = a * (b * c)$, where $a, b, c \in Q$

Hence, the operation $*$ is associative.

(vi) On Q , the binary operation $*$ is defined as $a * b = ab^2$

It can be found that

$$\frac{1}{2} * \frac{1}{3} = \frac{1}{2} \cdot \left(\frac{1}{3}\right)^2 = \frac{1}{2} \cdot \frac{1}{9} = \frac{1}{18}$$

and

$$\frac{1}{3} * \frac{1}{2} = \frac{1}{3} \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$$

$$\therefore \frac{1}{2} * \frac{1}{3} \neq \frac{1}{3} * \frac{1}{2}, \text{ where } \frac{1}{2} \text{ and } \frac{1}{3} \in Q$$

Hence, the operation $*$ is not commutative.

It is also seen that

$$\left(\frac{1}{2} * \frac{1}{3}\right) * \frac{1}{4} = \left[\frac{1}{2} \left(\frac{1}{3}\right)^2\right] * \frac{1}{4} = \frac{1}{18} * \frac{1}{4} = \frac{1}{18} \cdot \left(\frac{1}{4}\right)^2 = \frac{1}{18 \times 16} = \frac{1}{288}$$

and

$$\frac{1}{2} * \left(\frac{1}{3} * \frac{1}{4}\right) = \frac{1}{2} * \left[\frac{1}{3} \left(\frac{1}{4}\right)^2\right] = \frac{1}{2} * \frac{1}{48} = \frac{1}{2} \cdot \left(\frac{1}{48}\right)^2 = \frac{1}{2 \times 2304} = \frac{1}{4608}$$

$$\therefore \left(\frac{1}{2} * \frac{1}{3}\right) * \frac{1}{4} \neq \frac{1}{2} * \left(\frac{1}{3} * \frac{1}{4}\right), \text{ where } \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \in Q$$

Hence, the operation $*$ is not associative.

Hence, the operations defined in (ii), (iv), (v) are commutative and the operation defined in (v) is associative.

10. Find which of the operations given above has identity.

Solution:

An element $e \in Q$ will be the identity element for the binary operation $*$ if $a * e = a = e * a$, for all $a \in Q$.

However, there is no such element $e \in Q$ with respect to any of the six operations satisfying the above condition.

Hence, none of the six operations has identity.

11. Let $A = N \times N$ and $*$ be the binary operation on A defined by

$(a, b) * (c, d) = (a + c, b + d)$. Show that $*$ is commutative and associative. Find the identity element for $*$ on A , if any.

Solution:

Given:

$A = N \times N$ and $*$ is a binary operation on A and is defined by

$$(a, b) * (c, d) = (a + c, b + d)$$

Suppose $(a, b), (c, d) \in A$

Then, $a, b, c, d \in N$

We have:

$$(a, b) * (c, d) = (a + c, b + d)$$

$$(c, d) * (a, b) = (c + a, d + b) = (a + c, b + d)$$

[Since, addition is commutative in the set of natural numbers]

$$\therefore (a, b) * (c, d) = (c, d) * (a, b)$$

Hence, the operation $*$ is commutative.

Now, suppose $(a, b), (c, d), (e, f) \in A$

Then, $a, b, c, d, e, f \in N$

We have

$$[(a, b) * (c, d)] * (e, f) = (a + c, b + d) * (e, f) = (a + c + e, b + d + f)$$

and

$$(a, b) * [(c, d) * (e, f)] = (a, b) * (c + e, d + f) = (a + c + e, b + d + f)$$

$$\therefore [(a, b) * (c, d)] * (e, f) = (a, b) * [(c, d) * (e, f)]$$

Hence, the operation $*$ is associative.

Suppose an element $e = (e_1, e_2) \in A$ will be an identity element for the operation $*$ if $a * e = a = e * a$ for all $a = (a_1, a_2) \in A$

$$\text{i.e., } (a_1 + e_1, a_2 + e_2) = (a_1, a_2) = (e_1 + a_1, e_2 + a_2)$$

Which is not true for any element in A .

Hence, the operation $*$ does not have any identity element.

12. State whether the following statements are true or false. Justify.

(i) For an arbitrary binary operation $*$ on a set N , $a * a = a \forall a \in N$.

(ii) If $*$ is a commutative binary operation on N , then $a * (b * c) = (c * b) * a$

Solution:

(i) Defining an operation $*$ on N as $a * b = a + b \forall a, b \in N$

Then, in particular, for $b = a = 3$, we have

$$3 * 3 = 3 + 3 = 6 \neq 3$$

Thus, statement (i) is false.

(ii) R.H.S. = $(c * b) * a$

$$= (b * c) * a \text{ [Since, } * \text{ is commutative]}$$

$$= a * (b * c) \text{ [Again, as } * \text{ is commutative]}$$

$$= \text{L. H. S.}$$

$$\therefore a * (b * c) = (c * b) * a$$

Hence, statement (ii) is true.

13. Consider a binary operation $*$ on N defined as $a * b = a^3 + b^3$. Choose the correct answer.

(A) Is $*$ both associative and commutative?

(B) Is $*$ commutative but not associative?

(C) Is $*$ associative but not commutative?

(D) Is $*$ neither commutative nor associative?

Solution:

On N , the binary operation $*$ is defined as $a * b = a^3 + b^3$.

For, $a, b \in N$, we have

$$a * b = a^3 + b^3 = b^3 + a^3 = b * a \quad [\text{Since, addition is commutative in } N]$$

Hence, the operation $*$ is commutative.

It is also observed that

$$(1 * 2) * 3 = (1^3 + 2^3) * 3 = (1 + 8) * 3 = 9 * 3 = 9^3 + 3^3 = 729 + 27 = 756 \text{ and}$$

$$1 * (2 * 3) = 1 * (2^3 + 3^3) = 1 * (8 + 27) = 1 * 35 = 1^3 + 35^3 = 1 + 42875 = 42876$$

$$\therefore (1 * 2) * 3 \neq 1 * (2 * 3), \text{ where } 1, 2, 3 \in N$$

Hence, the operation $*$ is not associative.

Hence, the operation $*$ is commutative, but not associative.

Hence, the correct answer is B.

Miscellaneous Exercise on Chapter 1

1. Let $f: R \rightarrow R$ be defined as $f(x) = 10x + 7$. Find the function $g: R \rightarrow R$ such that $g \circ f = f \circ g = I_R$.

Solution:

Given that $f: R \rightarrow R$ is defined as $f(x) = 10x + 7$.

For one – one

Suppose $f(x) = f(y)$, where $x, y \in R$.

$$\Rightarrow 10x + 7 = 10y + 7$$

$$\Rightarrow x = y$$

Hence, f is a one-one function.

For onto

For $y \in R$, suppose $y = 10x + 7$.

$$\Rightarrow x = \frac{y-7}{10} \in R$$

Hence, for any $y \in R$, there exists $x = \frac{y-7}{10} \in R$ such that

$$f(x) = f\left(\frac{y-7}{10}\right) = 10\left(\frac{y-7}{10}\right) + 7 = y - 7 + 7 = y$$

$\therefore f$ is onto function.

Hence, f is one – one and onto.

Hence, f is an invertible function.

Let us define $g: R \rightarrow R$ as $g(y) = \frac{y-7}{10}$

Now, we have

$$g \circ f(x) = g(f(x)) = g(10x + 7) = \frac{(10x+7)-7}{10} = \frac{10x}{10} = x$$

and

$$f \circ g(y) = f(g(y)) = f\left(\frac{y-7}{10}\right) = 10\left(\frac{y-7}{10}\right) + 7 = y - 7 + 7 = y$$

$\therefore g \circ f = I_R$ and $f \circ g = I_R$.

Hence, the required function $g: R \rightarrow R$ is defined as $g(y) = \frac{y-7}{10}$.

2. Let $f: W \rightarrow W$ be defined as $f(n) = n - 1$, if n is odd and $f(n) = n + 1$, if n is even. Show that f is invertible. Find the inverse of f . Here, W is the set of all whole numbers.

Solution:

Given that:

$$f: W \rightarrow W \text{ is defined as } f(n) = \begin{cases} n - 1, & \text{if } n \text{ is odd} \\ n + 1, & \text{if } n \text{ is even} \end{cases}$$

For one-one

Suppose $f(n) = f(m)$.

Here it can be observed that if n is odd and m is even, then we will have $n - 1 = m + 1$.

$$\Rightarrow n - m = 2$$

However, this is impossible.

Similarly, the possibility of n being even and m being odd can also be ignored in a similar way.

Hence, both n and m must be either odd or even. Now, if both n and m are odd,

Then, we have

$$f(n) = f(m)$$

$$\Rightarrow n - 1 = m - 1$$

$$\Rightarrow n = m$$

Again, if both n and m are even,

Then, we have

$$f(n) = f(m)$$

$$\Rightarrow n + 1 = m + 1$$

$$\Rightarrow n = m$$

$\therefore f$ is one-one.

For onto

Here, it is clear that any odd number $2r + 1$ in co-domain N is the image of $2r$ in domain N

Again, any even number $2r$ in co-domain N is the image of $2r + 1$ in domain N .

$\therefore f$ is onto function.

Hence, f is an invertible function.

Let us define $g: W \rightarrow W$ as $g(m) = \begin{cases} m + 1, & \text{if } m \text{ is even} \\ m - 1, & \text{if } m \text{ is odd} \end{cases}$

Now, when n is odd

$$g \circ f(n) = g(f(n)) = g(n - 1) = n - 1 + 1 = n \text{ and}$$

When n is even

$$g \circ f(n) = g(f(n)) = g(n + 1) = n + 1 - 1 = n$$

Similarly,

When m is odd

$$f \circ g(m) = f(g(m)) = f(m - 1) = m - 1 + 1 = m \text{ and}$$

When m is even

$$f \circ g(m) = f(g(m)) = f(m + 1) = m + 1 - 1 = m$$

$$\therefore g \circ f = I_W \text{ and } f \circ g = I_W$$

Hence, f is invertible and the inverse of f is given by $f^{-1} = g$, which is the same as f .
Hence, the inverse of f is f itself.

3. If $f: R \rightarrow R$ is defined by $f(x) = x^2 - 3x + 2$, find $f(f(x))$.

Solution:

Given that $f: R \rightarrow R$ is defined as $f(x) = x^2 - 3x + 2$.

$$\text{Hence, } f(f(x)) = f(x^2 - 3x + 2)$$

$$= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$$

$$= (x^4 + 9x^2 + 4 - 6x^3 - 12x + 4x^2) + (-3x^2 + 9x - 6) + 2$$

$$= x^4 - 6x^3 + 10x^2 - 3x$$

4. Show that the function $f: R \rightarrow \{x \in R : -1 < x < 1\}$ defined by $f(x) = \frac{x}{1+|x|}$, $x \in R$ is one-one and onto function.

Solution:

Given that $f: R \rightarrow \{x \in R : -1 < x < 1\}$ is defined as $f(x) = \frac{x}{1+|x|}$, $x \in R$.

For one-one

Suppose $f(x) = f(y)$, where $x, y \in R$.

$$\Rightarrow \frac{x}{1+|x|} = \frac{y}{1+|y|}$$

It can be observed that if x is positive and y is negative,

Then, we have

$$\frac{x}{1+x} = \frac{y}{1-y} \Rightarrow 2xy = x - y$$

Since, x is positive and y is negative

$$x > y \Rightarrow x - y > 0$$

But, $2xy$ is negative.

Hence, $2xy \neq x - y$

Hence, the case of x being positive and y being negative can be ruled out.

Using a similar argument, x being negative and y being positive can also be ruled out.

$\therefore x$ and y have to be either positive or negative.

When x and y are both are positive, we have

$$f(x) = f(y) \Rightarrow \frac{x}{1+x} = \frac{y}{1+y} \Rightarrow x + xy = y + xy \Rightarrow x = y$$

When x and y are both are negative, we have

$$f(x) = f(y) \Rightarrow \frac{x}{1-x} = \frac{y}{1-y} \Rightarrow x - xy = y - xy \Rightarrow x = y$$

$\therefore f$ is one-one.

For onto

Now, suppose $y \in R$ such that $-1 < y < 1$.

If y is negative, then, there exists $x = \frac{y}{1+y} \in R$ such that

$$f(x) = f\left(\frac{y}{1+y}\right) = \frac{\left(\frac{y}{1+y}\right)}{1+\left|\frac{y}{1+y}\right|} = \frac{\frac{y}{1+y}}{1+\left(\frac{-y}{1+y}\right)} = \frac{y}{1+y-y} = y$$

If y is positive, then, there exists $x = \frac{y}{1-y} \in R$ such that

$$f(x) = f\left(\frac{y}{1-y}\right) = \frac{\left(\frac{y}{1-y}\right)}{1+\left|\frac{y}{1-y}\right|} = \frac{\frac{y}{1-y}}{1+\left(\frac{y}{1-y}\right)} = \frac{y}{1-y+y} = y$$

$\therefore f$ is onto function.

Hence, f is one-one and onto.

5. Show that the function $f: R \rightarrow R$ given by $f(x) = x^3$ is injective.

Solution:

We have: $f: R \rightarrow R$ is given as $f(x) = x^3$.

For one-one

Suppose $f(x) = f(y)$, where $x, y \in R$.

$$\Rightarrow x^3 = y^3 \dots (1)$$

Now, we need to show that $x = y$.

Suppose $x \neq y$, their cubes will also not be equal.

$$\Rightarrow x^3 \neq y^3$$

However, this is a contradiction to (1).

$$\therefore x = y$$

Hence, f is injective.

6. Give examples of two functions $f: N \rightarrow Z$ and $g: Z \rightarrow Z$ such that gof is injective but g is not injective.

(Hint: Consider $f(x) = x$ and $g(x) = |x|$).

Solution:

Define $f: N \rightarrow Z$ as $f(x) = x$ and $g: Z \rightarrow Z$ as $g(x) = |x|$.

We will now first show that g is not injective.

It can be seen that

$$g(-1) = |-1| = 1$$

$$g(1) = |1| = 1$$

$$\therefore g(-1) = g(1), \text{ but } -1 \neq 1.$$

$\therefore g$ is not injective.

Now, $g \circ f: N \rightarrow Z$ is defined as $g \circ f(x) = g(f(x)) = g(x) = |x|$.

Suppose $x, y \in N$ such that $g \circ f(x) = g \circ f(y)$.

$$\Rightarrow |x| = |y|$$

Since x and $y \in N$, both are positive.

$$\therefore |x| = |y| \Rightarrow x = y$$

Hence, $g \circ f$ is injective.

7. Give examples of two functions $f: N \rightarrow N$ and $g: N \rightarrow N$ such that $g \circ f$ is onto but f is not onto.

(Hint: Consider $f(x) = x + 1$ and $g(x) = \begin{cases} x - 1, & \text{if } x > 1 \\ 1, & \text{if } x = 1 \end{cases}$)

Solution:

Define $f: N \rightarrow N$ by $f(x) = x + 1$

and $g: N \rightarrow N$ by $g(x) = \begin{cases} x - 1, & \text{if } x > 1 \\ 1, & \text{if } x = 1 \end{cases}$

We will first show that f is not onto.

For this, consider element 1 in co-domain N . It is clear that this element is not an image of any of the elements in domain N .

Therefore, f is not onto function.

Now, $g \circ f: N \rightarrow N$ is defined by

$$g \circ f(x) = g(f(x)) = g(x + 1) = x + 1 - 1 = x \quad [x \in N \Rightarrow x + 1 > 1]$$

Hence, it is clear that for $y \in N$, there exists $x = y \in N$ such that $g \circ f(x) = y$.

Hence, $g \circ f$ is onto function.

8. Given a non-empty set X , consider $P(X)$ which is the set of all subsets of X .

Define the relation R in $P(X)$ as follows:

For subsets A, B in $P(X)$, ARB if and only if $A \subset B$. Is R an equivalence relation on $P(X)$?

Justify your answer.

Solution:

As we know that, every set is a subset of itself, ARA for all $A \in P(X)$.

$\therefore R$ is reflexive.

Suppose $ARB \Rightarrow A \subset B$.

This cannot be implied to $B \subset A$.

For instance, if $A = \{1, 2\}$ and $B = \{1, 2, 3\}$, then it is observed that B is not related to A .

$\therefore R$ is not symmetric.

Again, if ARB and BRC , then $A \subset B$ and $B \subset C$.

$\Rightarrow A \subset C$

$\Rightarrow ARC$

$\therefore R$ is transitive.

Hence, R is not an equivalence relation as it is not symmetric.

9. Given a non-empty set X , consider the binary operation $*$: $P(X) \times P(X) \rightarrow P(X)$ given by

$A * B = A \cap B \forall A, B$ in $P(X)$, where $P(X)$ is the power set of X . Show that X is the identity element for this operation and X is the only invertible element in $P(X)$ with respect to the operation $*$.

Solution:

Given the binary operation $*$:

$P(X) \times P(X) \rightarrow P(X)$ given by $A * B = A \cap B \forall A, B$ in $P(X)$

We know that $A \cap X = A = X \cap A$ for all $A \in P(X)$

$\Rightarrow A * X = A = X * A$ for all $A \in P(X)$

Hence, X is the identity element for the given binary operation $*$.

Now, an element $A \in P(X)$ is invertible if there exists $B \in P(X)$ such that

$A * B = X = B * A$ [As X is the identity element]

or

$A \cap B = X = B \cap A$

This case is possible only when $A = X = B$.

Hence, X is the only invertible element in $P(X)$ with respect to the given operation $*$.

Hence, the given result is proved.

10. Find the number of all onto functions from the set $\{1, 2, 3, \dots, n\}$ to itself.

Solution:

Onto functions from the set $\{1, 2, 3, \dots, n\}$ to itself is just a permutation on n symbols $1, 2, \dots, n$.

Hence, the total number of onto maps from $\{1, 2, \dots, n\}$ to itself is the same as the total number of permutations on n symbols $1, 2, \dots, n$, which is $n!$.

11. Let $S = \{a, b, c\}$ and $T = \{1, 2, 3\}$. Find F^{-1} of the following functions F from S to T , if it exists.

(i) $F = \{(a, 3), (b, 2), (c, 1)\}$

(ii) $F = \{(a, 2), (b, 1), (c, 1)\}$

Solution:

Given, $S = \{a, b, c\}, T = \{1, 2, 3\}$

(i) $F: S \rightarrow T$ has been defined as $F = \{(a, 3), (b, 2), (c, 1)\}$

$\Rightarrow F(a) = 3, F(b) = 2, F(c) = 1$

We can see that F is one-one and onto. Hence, inverse of F exists.

Hence, $F^{-1}: T \rightarrow S$ is given by $F^{-1} = \{(3, a), (2, b), (1, c)\}$.

(ii) $F: S \rightarrow T$ has been defined as $F = \{(a, 2), (b, 1), (c, 1)\}$

Since $F(b) = F(c) = 1$. Hence, F is not one-one function.

Hence, F is not invertible i.e., F^{-1} does not exist.

12. Consider the binary operations $*$: $R \times R \rightarrow R$ and o : $R \times R \rightarrow R$ defined as $a * b = |a - b|$ and $a o b = a, \forall a, b \in R$. Show that $*$ is commutative but not associative, o is associative but not commutative. Further, show that $\forall a, b, c \in R, a * (b o c) = (a * b) o (a * c)$. [If it is so, we say that the operation $*$ distributes over the operation o]. Does o distribute over $*$? Justify your answer.

Solution:

Given that $*$: $R \times R \rightarrow R$ and o : $R \times R \rightarrow R$ is defined as $a * b = |a - b|$ and

$a o b = a, \forall a, b \in R$

For $a, b \in R$, we have

$a * b = |a - b|$ and $b * a = |b - a| = |-(a - b)| = |a - b|$

$\therefore a * b = b * a$

Hence, the operation $*$ is commutative.

It is also observed that

$(1 * 2) * 3 = (|1 - 2|) * 3 = 1 * 3 = |1 - 3| = 2$

and

$1 * (2 * 3) = 1 * (|2 - 3|) = 1 * 1 = |1 - 1| = 0$

$\therefore (1 * 2) * 3 \neq 1 * (2 * 3)$ where $1, 2, 3 \in R$.

Hence, the operation $*$ is not associative.

Again, consider the operation o

It is observed that $1 o 2 = 1$ and $2 o 1 = 2$.

$\therefore 1 o 2 \neq 2 o 1$ where $1, 2 \in R$.

Hence, the operation o is not commutative.

Suppose $a, b, c \in R$. Then, we have

$$(a o b) o c = a o c = a$$

and

$$a o (b o c) = a o b = a$$

$\therefore a o b) o c = a o (b o c)$, where $a, b, c \in R$

Hence, the operation o is associative.

Now, suppose $a, b, c \in R$, then we have

$$a * (b o c) = a * b = |a - b|$$

$$(a * b) o (a * c) = (|a - b|) o (|a - c|) = |a - b|$$

Hence, $a * (b o c) = (a * b) o (a * c)$.

Hence, the operation $*$ distribute over o .

Now,

$$1 o (2 * 3) = 1 o (|2 - 3|) = 1 o 1 = 1$$

$$(1 o 2) * (1 o 3) = 1 * 1 = |1 - 1| = 0$$

$\therefore 1 o (2 * 3) \neq (1 o 2) * (1 o 3)$ where $1, 2, 3 \in R$

Hence, the operation o does not distribute over $*$.

13. Given a non-empty set X , let $*$: $P(X) \times P(X) \rightarrow P(X)$ be defined as $A * B = (A - B) \cup (B - A)$, $\forall A, B \in P(X)$. Show that the empty set ϕ is the identity for the operation $*$ and all the elements A of $P(X)$ are invertible with $A^{-1} = A$. (Hint: $(A - \phi) \cup (\phi - A) = A$ and $(A - A) \cup (A - A) = A * A = \phi$).

Solution:

Given that $*$: $P(X) \times P(X) \rightarrow P(X)$ is defined as $A * B = (A - B) \cup (B - A) \forall A, B \in P(X)$.

Suppose $A \in P(X)$. Then, we have

$$A * \phi = (A - \phi) \cup (\phi - A) = A \cup \phi = A$$

$$\phi * A = (\phi - A) \cup (A - \phi) = \phi \cup A = A$$

$$\therefore A * \phi = A = \phi * A \text{ for all } A \in P(X)$$

Thus, ϕ is the identity element for the given operation $*$.

Now, again an element $A \in P(X)$ will be invertible if there exists $B \in P(X)$ such that

$$A * B = \phi = B * A. \text{ [As } \phi \text{ is the identity element]}$$

Now, we have observed that

$$A * A = (A - A) \cup (A - A) = \phi \cup \phi = \phi \text{ for all } A \in P(X).$$

Hence, all the elements A of $P(X)$ are invertible with $A^{-1} = A$.

14. Define a binary operation $*$ on the set $\{0, 1, 2, 3, 4, 5\}$ as

$$a * b = \{a + b, \text{ if } a + b < 6$$

$$a + b - 6 \text{ if } a + b \geq 6 \}$$

Show that zero is the identity for this operation and each element $a \neq 0$ of the set is invertible with $6 - a$ being the inverse of a .

Solution:

Given, $X = \{0, 1, 2, 3, 4, 5\}$.

Here the operation $*$ on X is defined as $a * b = \{a + b, \text{ if } a + b < 6$

$$a + b - 6, \quad \text{if } a + b \geq 6 \}$$

An element $e \in X$ is the identity element for the operation $*$, if

$$a * e = a = e * a \text{ for all } a \in X$$

For $a \in X$, we have

$$a * 0 = a + 0 = a \quad [a \in X \Rightarrow a + 0 < 6]$$

$$0 * a = 0 + a = a \quad [a \in X \Rightarrow 0 + a < 6]$$

$$\therefore a * 0 = a = 0 * a \text{ for all } a \in X$$

Hence, 0 is the identity element for the given operation $*$.

An element $a \in X$ is invertible if there exists $b \in X$ such that $a * b = 0 = b * a$.

i.e., $a + b = 0 = b + a$, if $a + b < 6$ and

$$a + b - 6 = 0 = b + a - 6, \text{ if } a + b \geq 6$$

$$\Rightarrow a = -b \text{ or } b = 6 - a$$

But, $X = \{0, 1, 2, 3, 4, 5\}$ and $a, b \in X$

Hence, $a \neq -b$.

$\therefore b = 6 - a$ is the inverse of a for all $a \in X$.

Hence, the inverse of an element $a \in X, a \neq 0$ is $6 - a$ i.e., $a^{-1} = 6 - a$.

15. Let $A = \{-1, 0, 1, 2\}$, $B = \{-4, -2, 0, 2\}$ and $f, g: A \rightarrow B$ be functions defined by

$f(x) = x^2 - x, x \in A$ and $g(x) = 2 \left| x - \frac{1}{2} \right| - 1, x \in A$. Are f and g equal? Justify your answer. (Hint: One may note that two functions $f: A \rightarrow B$ and $g: A \rightarrow B$ such that $f(a) = g(a) \forall a \in A$, are called equal functions).

Solution:

Given that $A = \{-1, 0, 1, 2\}, B = \{-4, -2, 0, 2\}$.

Also, it is given that $f, g: A \rightarrow B$ are defined by

$$f(x) = x^2 - x, x \in A \text{ and } g(x) = 2 \left| x - \frac{1}{2} \right| - 1, x \in A$$

It is observed that

$$f(-1) = (-1)^2 - (-1) = 1 + 1 = 2$$

$$\text{and } g(-1) = 2 \left| (-1) - \frac{1}{2} \right| - 1 = 2 \left(\frac{3}{2} \right) - 1 = 3 - 1 = 2$$

$$\Rightarrow f(-1) = g(-1)$$

$$f(0) = (0)^2 - (0) = 0$$

$$\text{and } g(0) = 2 \left| 0 - \frac{1}{2} \right| - 1 = 2 \left(\frac{1}{2} \right) - 1 = 1 - 1 = 0$$

$$\Rightarrow f(0) = g(0)$$

$$f(1) = (1)^2 - (1) = 1 - 1 = 0$$

$$\text{and } g(1) = 2 \left| (1) - \frac{1}{2} \right| - 1 = 2 \left(\frac{1}{2} \right) - 1 = 1 - 1 = 0$$

$$\Rightarrow f(1) = g(1)$$

$$f(2) = (2)^2 - (2) = 4 - 2 = 2$$

$$\text{And } g(2) = 2 \left| (2) - \frac{1}{2} \right| - 1 = 2 \left(\frac{3}{2} \right) - 1 = 3 - 1 = 2$$

$$\Rightarrow f(2) = g(2)$$

$$\therefore f(a) = g(a) \text{ for all } a \in A$$

Hence, the functions f and g are equal.

16. Let $A = \{1, 2, 3\}$. Then number of relations containing $(1, 2)$ and $(1, 3)$ which are reflexive and symmetric but not transitive is

- (A) 1
- (B) 2
- (C) 3
- (D) 4

Solution:

The given set is $A = \{1, 2, 3\}$.

The smallest relation containing $(1, 2)$ and $(1, 3)$ which is reflexive and symmetric, but not transitive is given by:

$$R = \{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 1), (3, 1)\}$$

This is because relation R is reflexive as $(1, 1), (2, 2), (3, 3) \in R$.

Relation R is symmetric since $(1, 2), (2, 1) \in R$ and $(1, 3), (3, 1) \in R$

But relation R is not transitive as $(3, 1), (1, 2) \in R$, but $(3, 2) \notin R$.

Now we can observe that adding any element to it will make the relation transitive or not symmetric.

Hence, the total number of desired relations is one.

The correct answer is A.

17. Let $A = \{1, 2, 3\}$. Then number of equivalence relations containing $(1, 2)$ is

- (A) 1
- (B) 2
- (C) 3
- (D) 4

Solution:

It is given that $A = \{1, 2, 3\}$.

The smallest equivalence relation containing $(1, 2)$ is given by,

$$R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$$

Now, we are left with only four pairs i.e., $(2, 3), (3, 2), (1, 3)$, and $(3, 1)$.

If we add any ordered pair [say $(2, 3)$] to R_1 , then for symmetry, we must add $(3, 2)$. Also, for transitivity we need to add $(1, 3)$ and $(3, 1)$.

Hence, the only equivalence relation (bigger than R_1) is the universal relation.

Hence, the total number of equivalence relations containing $(1, 2)$ is two.

The correct answer is B.

18. Let $f: R \rightarrow R$ be the Signum Function defined as

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

and $g: R \rightarrow R$ be the Greatest Integer Function given by $g(x) = [x]$, where $[x]$ is greatest integer less than or equal to x . Then, does $f \circ g$ and $g \circ f$ coincide in $(0, 1]$?

Solution:

Given that,

$$f: R \rightarrow R \text{ is defined as } f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

Again, $g: R \rightarrow R$ is defined as $g(x) = [x]$, where $[x]$ is the greatest integer less than or equal to x .

Now, suppose $x \in (0, 1]$.

Then, we have

$$[x] = 1 \text{ if } x = 1 \text{ and } [x] = 0 \text{ if } 0 < x < 1.$$

$$\begin{aligned} f \circ g(x) &= f(g(x)) = f([x]) = \begin{cases} f(1), & \text{if } x = 1 \\ f(0), & \text{if } x \in (0, 1) \end{cases} \\ &= \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{if } x \in (0, 1) \end{cases} \end{aligned}$$

$$g \circ f(x) = g(f(x)) = g(1) \text{ [as } x > 0 \text{]}$$

$$= [1] = 1$$

Hence, when $x \in (0, 1)$, we have $f \circ g(x) = 0$ and $g \circ f(x) = 1$.

Hence, $f \circ g$ and $g \circ f$ do not coincide in $(0, 1]$.

19. Number of binary operations on the set $\{a, b\}$ are

- (A) 10
- (B) 16
- (C) 20
- (D) 8

Solution:

As we know that a binary operation $*$ on $\{a, b\}$ is a function from $\{a, b\} \times \{a, b\} \rightarrow \{a, b\}$

i.e., $*$ is a function from $\{(a, a), (a, b), (b, a), (b, b)\} \rightarrow \{a, b\}$.

Hence, every element of $\{a, b\} \times \{a, b\}$ has two options a or b .

Hence, the total number of binary operations on the set $\{a, b\}$ is 2^4 i.e., 16.

The correct answer is B .