CBSE NCERT Solutions for Class 12 Maths Chapter 11

Back of Chapter Questions

Exercise 11. 1

1. If a line makes angles $90^o, 135^o, 45^o$ with the x, y and z-axes respectively, find its direction cosines.

Solution:

Considering direction cosines of the line as l, m, and n.

$l = \cos 90^o = 0$

$m = \cos 135^o = -\frac{1}{\sqrt{2}}$

$n = \cos 45^o = \frac{1}{\sqrt{2}}$

Hence, the direction cosines of the line are $0, -\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$.

2. Find the direction cosines of a line which makes equal angles with the coordinate axes.

Solution:

Consider, the direction cosine of the line making an angle α with each of the coordinate axes.

$\therefore l = \cos\alpha, m = \cos\alpha, n = \cos\alpha$

$l^2 + m^2 + n^2 = 1$

$\Rightarrow \cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$

$\Rightarrow 3\cos^2\alpha = 1$

$\Rightarrow \cos^2\alpha = \frac{1}{3}$

$\Rightarrow \cos\alpha = \pm\frac{1}{\sqrt{3}}$

Therefore, the direction cosines of the line, which is equally inclined to the coordinate axes, are $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$ and $\pm\frac{1}{\sqrt{3}}$.

3. If a line has the direction ratios $-18, 12, -4$, then what are its direction cosines?

Solution:

Considering direction cosines of the line as l, m, and n.

The line has direction ratios of $-18, 12$ and -4 then its direction cosines are

$\mathrm{l} = \frac{-18}{\sqrt{(-18)^2+(12)^2+(-4)^2}}$,

$\mathrm{m} = \frac{12}{\sqrt{(-18)^2+(12)^2+(-4)^2}}$

$\mathrm{n} = \frac{-4}{\sqrt{(-18)^2+(12)^2+(-4)^2}}$

Therefore, $\mathrm{l} = \frac{-18}{22}$

$\mathrm{m} = \frac{12}{22}$,

$\mathrm{n} = \frac{-4}{22}$

Thus, the direction cosines are $-\frac{9}{11}, \frac{6}{11}$ and $\frac{-2}{11}$

4. Show that the points $(2, 3, 4), (-1, -2, 1), (5, 8, 7)$ are collinear.

Solution:

Given: $A(2, 3, 4), B(-1, -2, 1)$ and $C(5, 8, 7)$.

We know that the direction ratios of line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are given by $x_2 - x_1, y_2 - y_1$ and $z_2 - z_1$.

The direction ratios of AB are $(-1-2), (-2-3)$ and $(1-4)$

=> $-3, -5$ and -3.

The direction ratios of BC are $(5-(-1)), (8-(-2))$ and $(7-1)$.

=> $6, 10$ and 6.

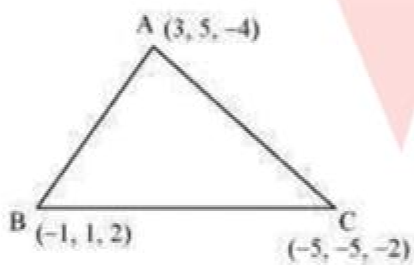
It is seen that the direction ratios of BC are -2 times that of AB therefore, they are proportional.

Thus, AB is parallel to BC. Since point B is common to both AB and BC, points A, B and C are collinear.

5. Find the direction cosines of the sides of the triangle whose vertices are $(3, 5, -4), (-1, 1, 2)$ and $(-5, -5, -2)$

Solution:

Given that the vertices of ΔABC are $A(3, 5, -4), B(-1, 1, 2)$ and $C(-5, -5, -2)$.



We know that the direction ratios of line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are given by $x_2 - x_1, y_2 - y_1$ and $z_2 - z_1$

The direction ratios of side AB with the points, $A(3, 5, -4), B(-1, 1, 2)$ are:

=> $(-1-3), (1-5)$ and $(2-(-4))$

=> $-4, -4$ and 6.

Then $\sqrt{(-4)^2 + (-4)^2 + (6)^2} = \sqrt{16+16+36}$

$= \sqrt{68}$

$= 2\sqrt{17}$

Therefore, the direction cosines of AB are

=> $\frac{-4}{\sqrt{(-4)^2+(-4)^2+(6)^2}}, \frac{-4}{\sqrt{(-4)^2+(-4)^2+(6)^2}}, \frac{6}{\sqrt{(-4)^2+(-4)^2+(6)^2}}$

=> $\frac{-4}{2\sqrt{17}}, -\frac{4}{2\sqrt{17}}, \frac{6}{2\sqrt{17}}$

=> $\frac{-2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}$.

The direction ratios of BC with the points, $B(-1, 1, 2)$ and $C(-5, -5, -2)$ are:

=> $(-5-(-1)), (-5-1)$ and $(-2-2)$

=> $-4, -6$ and -4.

Therefore, the direction cosines of BC are

=> $\frac{-4}{\sqrt{(-4)^2+(-6)^2+(-4)^2}}, \frac{-6}{\sqrt{(-4)^2+(-6)^2+(-4)^2}}, \frac{-4}{\sqrt{(-4)^2+(-6)^2+(-4)^2}}$

=> $\frac{-4}{2\sqrt{17}}, \frac{-6}{2\sqrt{17}}, \frac{-4}{2\sqrt{17}}$

The direction ratios of CA with the points, $A(3, 5, -4)$ and $C(-5, -5, -2)$ are:

=> $(-5-3), (-5-5)$ and $(-2-(-4))$

=> $-8, -10$ and 2.

Therefore, the direction cosines of AC are

=> $\frac{-8}{\sqrt{(-8)^2+(10)^2+(2)^2}}, \frac{-10}{\sqrt{(-8)^2+(10)^2+(2)^2}}, \frac{2}{\sqrt{(-8)^2+(10)^2+(2)^2}}$

=> $\frac{-8}{2\sqrt{42}}, \frac{-10}{2\sqrt{42}}, \frac{2}{2\sqrt{42}}$

Exercise 11.2

1. Show that the three lines with direction cosines

$\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}; \frac{4}{13}, \frac{12}{13}, \frac{3}{13}; \frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ are mutually perpendicular.

Solution:

We know that two lines with direction cosines, l_1, m_1, n_1 and l_2, m_2, n_2 are perpendicular to each other

When $l_1l_2 + m_1m_2 + n_1n_2 = 0$

(i) For the lines with direction cosines, $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$ and $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$

$l_1l_2 + m_1m_2 + n_1n_2 = \frac{12}{13} \times \frac{4}{13} + \left(\frac{-3}{13}\right) \times \frac{12}{13} + \left(\frac{-4}{13}\right) \times \frac{3}{13}$

$= \frac{48}{169} - \frac{36}{169} - \frac{12}{169}$

$= 0$

Thus, the lines are perpendicular.

(ii) For the lines with direction cosines $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$ and $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$

$l_1l_2 + m_1m_2 + n_1n_2 = \frac{4}{13} \times \frac{3}{13} + \frac{12}{13} \times \left(\frac{-4}{13}\right) + \frac{3}{13} \times \frac{12}{13}$

$= \frac{12}{169} - \frac{48}{169} + \frac{36}{169}$

$= 0$

Thus, the lines are perpendicular.

(iii) For the lines with direction cosines, $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ and $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$, we obtain

$l_1l_2 + m_1m_2 + n_1n_2 = \left(\frac{3}{13}\right) \times \left(\frac{12}{13}\right) + \left(\frac{-4}{13}\right) \times \left(\frac{-3}{13}\right) + \left(\frac{12}{13}\right) \times \left(\frac{-4}{13}\right)$

$= \frac{36}{169} + \frac{12}{169} - \frac{48}{169}$

$= 0$

Thus, the lines are perpendicular.

Thus, all the lines are mutually perpendicular.

2. Show that the line through the points $(1, -1, 2)(3, 4, -2)$ is perpendicular to the line through the points $(0, 3, 2)$ and $(3, 5, 6)$.

Solution:

Consider, AB as the line joining the points $(1, -1, 2)$ and $(3, 4, -2)$ and CD as the line joining the points $(0, 3, 2)$ and $(3, 5, 6)$.

The direction ratios a_1, b_1, c_1 of AB are $(3-1), (4-(-1))$ and $(-2-2)$

=> $2, 5$ and -4.

The direction ratios a_2, b_2, c_2 of CD are $(3-0), (5-3)$ and $(6-2)$.

=> $3, 2$ and 4.

AB and CD will be perpendicular to each other, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$a_1a_2 + b_1b_2 + c_1c_2 = 2 \times 3 + 5 \times 2 + (-4) \times 4$

$= 6 + 10 - 16$

$= 0$

Thus, AB and CD are perpendicular to each other.

3. Show that the line through the points $(4, 7, 8), (2,3, 4)$ is parallel to the line through the points $(-1, -2, 1), (1, 2, 5)$.

Solution:

Consider, AB, the line through the points $(4, 7, 8)$ and $(2, 3, 4)$ and CD be the line through the points $(-1, -2, 1)$ and $(1, 2, 5)$.

The directions ratios a_1, b_1, c_1 of AB are $(2-4), (3-7)$ and $(4-8)$

=> $-2, -4$ and -4.

The direction ratios, a_2, b_2, c_2 of CD are $(1-(-1)), (2-(-2))$ and $(5-1)$

=> $2, 4$ and 4.

AB will be parallel to CD, when $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$$\frac{a_1}{a_2} = \frac{-2}{2} = -1$$

$$\frac{b_1}{b_2} = \frac{-4}{4} = -1$$

$$\frac{c_1}{c_2} = \frac{-4}{4} = -1$$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Therefore, AB is parallel to CD.

4. Find the equation of the line which passes through the point $(1, 2, 3)$ and is parallel to the vector $3\hat{i} + 2\hat{j} - 2\hat{k}$

Solution:

Given that the line passes through the point $A(1, 2, 3)$.

Thus, the position vector through A is

$\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{b} = 3\hat{i} + 2\hat{j} - 2\hat{k}$

We know that the line which passes through point A and parallel to $\vec{b}$ is given by $\vec{r} = \vec{a} + \lambda\vec{b}$, where λ is a constant.

$\Rightarrow \vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 2\hat{k})$

This is the required equation of the line.

5. Find the equation of the line in vector and in Cartesian form that passes through the point with position vector $2\hat{i} - \hat{j} + 4\hat{k}$ and is in the direction $\hat{i} + 2\hat{j} - \hat{k}$

Solution:

Given that the line passes through the point with position vector

$\vec{a} = 2\hat{i} - \hat{j} + 4\hat{k}$ ...(i)

$\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ ...(ii)

We know that a line through a point with position vector $\vec{a}$ and parallel to $\vec{b}$ is given by the equation, $\vec{r} = \vec{a} + \lambda\vec{b}$

$\Rightarrow \vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda(\hat{i} + 2\hat{j} - \hat{k})$

This is the required equation of the line in vector form.

$\vec{r} = x\hat{i} - y\hat{j} + z\hat{k}$

$\Rightarrow x\hat{i} - y\hat{j} + z\hat{k} = (\lambda + 2)\hat{i} + (2\lambda - 1)\hat{j} + (-\lambda + 4)\hat{k}$

Eliminating λ, we obtain the Cartesian form equation as

$$\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$$

This is the required equation of the given line in Cartesian form.

6. Find the Cartesian equation of the line which passes through the point $(-2, 4, -5)$ and parallel to the line given by $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$

Solution:

Given that the line passes through the point $(-2, 4, -5)$ and is parallel to $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$

The direction ratios of the line $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$ are $3, 5$ and 6.

The required line is parallel to $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$

Therefore, its direction ratios are $3k, 5k$ and $6k$ where $k \neq 0$

We know that the equation of the line through the point (x_1, y_1, z_1) and with direction ratios a, b, c is given by $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

Therefore the equation of the required line is

$$\frac{x-(-2)}{3k} = \frac{y-4}{5k} = \frac{z-(-)5}{6k}$$

$\Rightarrow \frac{x+2}{3} = \frac{y-4}{5} = \frac{z+5}{6} = k$

7. The Cartesian equation of a line is $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$. Write its vector form.

Solution:

The Cartesian equation of the line is

$\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$ (i)

The given line passes through the point $(5,-4,6)$. [from (1)]

The position vector of this point is $\vec{a}=5\hat{i}-4\hat{j}+6\hat{k}$

The direction ratios of the given line are 3, 7 and 2. [from (1)]

This means that the line is in the direction of vector $\vec{b}=3\hat{i}+7\hat{j}+2\hat{k}$

We know that the line through position vector $\vec{a}$ and in the direction of the vector $\vec{b}$ is given by the equation $\vec{r}=\vec{a}+\lambda\vec{b},\ \lambda\in R$

$\Rightarrow \vec{r}=(5\hat{i}-4\hat{j}+6\hat{k})+\lambda(3\hat{i}+7\hat{j}+2\hat{k})$

This is the required equation of the given line in vector form.

8. Find the vector and the Cartesian equations of the lines that passes through the origin and $(5,-2,3)$.

Solution:

The required line passes through the origin. Therefore, its position vector is given by,

$\vec{a}=\vec{0}$ ...(i)

The direction ratios of the line through origin and $(5,-2,3)$ are:

$(5-0)=5$,

$(-2-0)=-2$,

$(3-0)=3$

The line is parallel to the vector given by the equation $\vec{b}=5\hat{i}-2\hat{j}+3\hat{k}$

The equation of the line in vector form through a point with position vector $\vec{a}$ and parallel to $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}, \lambda \in R$

$\Rightarrow \vec{r} = \vec{0} + \lambda(5\hat{i} - 2\hat{j} + 3\hat{k})$

$\Rightarrow \vec{r} = \lambda(5\hat{i} - 2\hat{j} + 3\hat{k})$

The equation of the line through the point (x_1, y_1, z_1) and direction ratios a, b, c is given by

$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

Thus, the equation of the required line in the Cartesian form is

$\frac{x-0}{5} = \frac{y-0}{-2} = \frac{z-0}{3}$

$\Rightarrow \frac{x}{5} = \frac{y}{-2} = \frac{z}{3}$

9. Find the vector and the Cartesian equations of the line that passes through the points $(3, -2, -5), (3, -2, 6)$.

Solution:

Consider the line passing through the points $P(3, -2, -5)$ and $Q(3, -2, 6)$ be PQ.

As PQ passes through $P(3, -2, -5)$, its position vector is given by

$\vec{a} = 3\hat{i} - 2\hat{j} - 5\hat{k}$

The direction ratios of PQ are given by,

$(3 - 3) = 0,$

$(-2 - (-2)) = 0,$

$(6 - (-5)) = 11$

Therefore, equation of the vector in the direction of PQ is

$\vec{b} = 0.\hat{i} + 0.\hat{j} + 11\hat{k} = 11\hat{k}$

Thus, the equation of PQ in vector form is given by $\vec{r} = \vec{a} + \lambda\vec{b}, \lambda \in R$

$= \vec{r} = (3\hat{i} - 2\hat{j} - 5\hat{k}) + 11\lambda\hat{k}$

The equation of PQ in Cartesian form is

$\Rightarrow \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

$\Rightarrow \frac{x-3}{0} = \frac{y+2}{0} = \frac{z+5}{11}$

Cartesian equations of the line that passes through the points is $\frac{x-3}{0} = \frac{y+2}{0} = \frac{z+5}{11}$

10. Find the angle between the following pairs of lines:

(i) $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$ and $\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$

(ii) $\vec{r} = 3\hat{i} + \hat{j} - 2\hat{k} + \lambda(\hat{i} - \hat{j} - 2\hat{k})$ and $\vec{r} = 2\hat{i} - \hat{j} - 56\hat{k} + \mu(3\hat{i} - 5\hat{j} - 4\hat{k})$

Solution:

(i) Consider Q as the angle between the given lines

The angle between the given pair of lines is given by, $\cos Q = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$

The given lines are parallel to the vectors, $\vec{b}_1 = 3\hat{i} + 2\hat{j} + 6\hat{k}$ and $\vec{b}_2 = \hat{i} + 2\hat{j} + 2\hat{k}$ respectively.

Therefore,

$|\vec{b}_1| = \sqrt{3^2 + 2^2 + 6^2} = 7$

$|\vec{b}_2| = \sqrt{(1)^2 + (2)^2 + (2)^2} = 3$

$\vec{b}_1 \cdot \vec{b}_2 = (3\hat{i} + 2\hat{j} + 6\hat{k}) \cdot (\hat{i} + 2\hat{j} + 2\hat{k})$

$= 3 \times 1 + 2 \times 2 + 6 \times 2$

$= 3 + 4 + 12$

$= 19$

$\Rightarrow \cos Q = \dfrac{19}{7 \times 3}$

$\Rightarrow Q = \cos^{-1}\left(\frac{19}{21}\right)$

(ii) The given lines are parallel to the vectors, $\vec{b}_1 = \hat{i} - \hat{j} - 2\hat{k}$ and $\vec{b}_2 = 3\hat{i} - 5\hat{j} - 4\hat{k}$ respectively.

$\therefore |\vec{b}_1| = \sqrt{(1)^2 + (-1)^2 + (-2)^2} = \sqrt{6}$

$|\vec{b}_2| = \sqrt{(3)^2 + (-5)^2 + (-4)^2} = \sqrt{50} = 5\sqrt{2}$

$\vec{b}_1 \cdot \vec{b}_2 = (\hat{i} - \hat{j} - 2\hat{k}) \cdot (3\hat{i} - 5\hat{j} - 4\hat{k})$

$= 1 \cdot 3 - 1(-5) - 2(-4)$

$= 3 + 5 + 8$

$= 16$

$$\cos Q = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$$

$$\Rightarrow \cos Q = \frac{16}{\sqrt{6} \cdot 5\sqrt{2}} = \frac{16}{\sqrt{2}.\sqrt{3} \cdot 5\sqrt{2}} = \frac{16}{10\sqrt{3}}$$

$$\Rightarrow \cos Q = \frac{8}{5\sqrt{3}}$$

$\Rightarrow Q = \cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$

11. Find the angle between the following pair of lines:

(i) $\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}$ and $\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$

(ii) $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ and $\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$

Solution:

(i) Let $\vec{b}_1$ and $\vec{b}_2$ be the vectors parallel to the pair of lines,

$\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}$ and $\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$

$\therefore\ \vec{b}_1 = 2\hat{i} + 5\hat{j} - 3\hat{k}$ and $\vec{b}_2 = -\hat{i} + 8\hat{j} + 4\hat{k}$

$|\vec{b}_1| = \sqrt{(2)^2 + (5)^2 + (-3)^2} = \sqrt{38}$

$|\vec{b}_2| = \sqrt{(-1)^2 + (8)^2 + (4)^2} = \sqrt{81} = 9$

$\vec{b}_1 \cdot \vec{b}_2 = (2\hat{i} + 5\hat{j} - 3\hat{k}) \cdot (-\hat{i} + 8\hat{j} + 4\hat{k})$

$= 2(-1) + 5 \times 8 + (-3) \cdot 4$

$= -2 + 40 - 12$

$= 26$

The angle Q between the given pair of lines is given by the relation,

$\cos Q = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$

$\Rightarrow \cos Q = \frac{26}{9\sqrt{38}}$

$\Rightarrow Q = \cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$

(ii) Let $\vec{b}_1, \vec{b}_2$ be the vectors parallel to the given pair of lines,

$\frac{x-5}{4} = \frac{y-5}{1} = \frac{z-3}{8}$ and $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ respectively.

$\vec{b}_1 = 2\hat{i} + 2\hat{j} + \hat{k}$

$\vec{b}_2 = 4\hat{i} + \hat{j} + 8\hat{k}$

$\therefore |\vec{b}_1| = \sqrt{(2)^2 + (2)^2 + (1)^2} = \sqrt{9} = 3$

$|\vec{b}_2| = \sqrt{4^2 + 1^2 + 8^2} = \sqrt{81} = 9$

$\vec{b}_1 \cdot \vec{b}_2 = (2\hat{i} + 2\hat{j} + \hat{k}) \cdot (4\hat{i} + \hat{j} + 8\hat{k})$

$= 2 \times 4 + 2 \times 1 + 1 \times 8$

$= 8 + 2 + 8$

$= 18$

If Q is the angle between the given pair of lines, then $\cos Q = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$

$\Rightarrow \cos Q = \frac{18}{3 \times 9} = \frac{2}{3}$

$\Rightarrow Q = \cos^{-1}\left(\frac{2}{3}\right)$

12. Find the values of p so the line $\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$ and $\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$ are at right angles.

Solution:

Given equations can be written in the standard form as

$\frac{x-1}{-3}=\frac{y-2}{\frac{2p}{7}}=\frac{z-3}{2}$ and $\frac{x-1}{\frac{-3p}{7}}=\frac{y-5}{1}=\frac{z-6}{-5}$

The direction ratios of the lines are $-3,\frac{2p}{7},2$ and $\frac{-3p}{7},1,-5$ respectively.

Two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2 are perpendicular to each other, if $a_1a_2+b_1b_2+c_1c_2=0$

$\therefore (-3).\left(\frac{-3p}{7}\right)+\left(\frac{2p}{7}\right).(1)+2.(-5)=0$

$\Rightarrow \frac{9p}{7}+\frac{2p}{7}=10$

$\Rightarrow 11p=70$

$\Rightarrow p=\frac{70}{11}$

Therefore, the value of p is $\frac{70}{11}$.

13. Show that the lines $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ are perpendicular to each other.

Solution:

Equations for the given lines are given as: $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$

Direction ratios for the respective lines are: $7,-5,1$ and $1,2,3$.

Two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2 are perpendicular to each other, When $a_1a_2+b_1b_2+c_1c_2=0$

$\therefore 7\times 1+(-5)\times 2+1\times 3$

$=7-10+3$

$=0$

Hence, the given lines are perpendicular to each other.

14. Find the shortest distance between the lines

$\vec{r} = (\hat{\imath} + 2\hat{\jmath} + \hat{k}) + \lambda(\hat{\imath} - \hat{\jmath} + \hat{k})$ and

$\vec{r} = 2\hat{\imath} - \hat{\jmath} - \hat{k} + \mu(2\hat{\imath} + \hat{\jmath} + 2\hat{k})$

Solution:

Equations for the given lines can be written as:

$\vec{r} = (\hat{\imath} + 2\hat{\jmath} + \hat{k}) + \lambda(\hat{\imath} - \hat{\jmath} + \hat{k})$

$\vec{r} = 2\hat{\imath} - \hat{\jmath} - \hat{k} + \mu(2\hat{\imath} + \hat{\jmath} + 2\hat{k})$

We know that the shortest distance between the lines, $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, is given by,

$d = \left|\frac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right|$ ...(i)

Comparing the given equations, we get

$\vec{a}_1 = \hat{\imath} + 2\hat{\jmath} + \hat{k}$

$\vec{b}_1 = \hat{\imath} - \hat{\jmath} + \hat{k}$

$\vec{a}_2 = 2\hat{\imath} - \hat{\jmath} - \hat{k}$

$\vec{b}_2 = 2\hat{\imath} + \hat{\jmath} + 2\hat{k}$

$\vec{a}_2 - \vec{a}_1 = (2\hat{\imath} - \hat{\jmath} - \hat{k}) - (\hat{\imath} + 2j + \hat{k}) = \hat{\imath} - 3\hat{\jmath} - 2\hat{k}$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$\vec{b}_1 \times \vec{b}_2 = (-2 - 1)\hat{\imath} - (2 - 2)\hat{\jmath} + (1 + 2)\hat{k} = -3\hat{\imath} + 3\hat{k}$

$\Rightarrow |\vec{b}_1 \times \vec{b}_2| = \sqrt{(-3)^2 + (3)^2} = \sqrt{9 + 9} = \sqrt{18} = 3\sqrt{2}$

Substituting all the values in the equation (i), we get

$d = \left|\frac{(-3\hat{\imath} + 3\hat{k})\cdot(\hat{\imath} - 3\hat{\jmath} - 2\hat{k})}{3\sqrt{2}}\right|$

$\Rightarrow d = \left|\frac{-3.1 + 3(-2)}{3\sqrt{2}}\right|$

$\Rightarrow d = \left|\frac{-9}{3\sqrt{2}}\right|$

$$\Rightarrow d = \frac{3}{\sqrt{2}} = \frac{3 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{3\sqrt{2}}{2}$$

Thus, the shortest distance between the two lines is $\frac{3\sqrt{2}}{2}$ units.

15. Find the shortest distance between the lines

$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

Solution:

Given lines are:

$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

We know that the shortest distance between the two lines is:

$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is given by,

$$d = \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}} \quad \ldots(i)$$

Comparing the given equations, we get

$x_1 = -1, \quad y_1 = -1, \quad z_1 = -1$

$a_1 = 7, \quad b_1 = -6, \quad c_1 = 1$

$x_2 = 3, \quad y_2 = 5, \quad z_2 = 7$

$a_2 = 1, \quad b_2 = -2, \quad c_2 = 1$

Therefore, $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$

$= 4(-6+2) - 6(7-1) + 8(-14+6)$

$= -16 - 36 - 64$

$= -116$

$\Rightarrow \sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}$

$= \sqrt{(-6+2)^2 + (1+7)^2 + (-14+6)^2}$

$= \sqrt{16 + 36 + 64}$

$= \sqrt{116}$

$= 2\sqrt{29}$

Substituting all the values in equation (i), we get

$$d = \frac{-116}{2\sqrt{29}} = \frac{-58}{\sqrt{29}} = \frac{-2 \times 29}{\sqrt{29}} = -2\sqrt{29}$$

As distance is always non-negative, the distance between the lines $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$ is $2\sqrt{29}$ units.

16. Find the shortest distance between the lines whose vector equations are

$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$

and $\vec{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \mu(2\hat{i} + 3\hat{j} + \hat{k})$

Solution:

Vector equations of the given lines are:

$\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$ and $\vec{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \mu(2\hat{i} + 3\hat{j} + \hat{k})$

We know that the shortest distance between the lines, $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$. The given lines are given by,

$d = \left|\frac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right|$ ...(i)

Comparing the given equations with $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$

$\vec{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{b}_1 = \hat{i} - 3\hat{j} + 2\hat{k}$

$\vec{a}_2 = 4\hat{i} + 5\hat{j} + 6\hat{k}$

$\vec{b}_2 = 2\hat{i} + 3\hat{j} + \hat{k}$

$\vec{a}_2 - \vec{a}_1 = (4\hat{i} + 5\hat{j} + 6\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 3\hat{i} + 3\hat{j} + 3\hat{k}$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix} = (-3-6)\hat{i} - (1-4)\hat{j} + (3+6)\hat{k} = -9\hat{i} + 3\hat{j} + 9\hat{k}$$

$$\Rightarrow |\vec{b}_1 \times \vec{b}_2| = \sqrt{(-9)^2 + (3)^2 + (9)^2} = \sqrt{81+9+81} = \sqrt{171} = 3\sqrt{19}$$

$$(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) = (-9\hat{i} + 3\hat{j} + 9\hat{k}) \cdot (3\hat{i} + 3\hat{j} + 3\hat{k})$$

$$= -9 \times 3 + 3 \times 3 + 9 \times 3$$

$$= 9$$

Substituting all the values in equation (i), we get

$$d = \left|\frac{9}{3\sqrt{19}}\right| = \frac{3}{\sqrt{19}}$$

Thus, the shortest distance between the two given lines is $\frac{3}{\sqrt{19}}$ units.

17. Find the shortest distance between the lines whose vector equations are

$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k}$ and

$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$

Solution:

The given vector equations are;

$$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k}$$
$$\Rightarrow \vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + t(-\hat{i} + \hat{j} - 2\hat{k}) \quad \ldots(i)$$

$$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$$

$$\Rightarrow \vec{r} = (\hat{i} - \hat{j} - \hat{k}) + s(\hat{i} + 2\hat{j} - 2\hat{k}) \quad \ldots(ii)$$

We know that the shortest distance between the lines, $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, is given by,

$$d = \left|\frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right| \quad \ldots(iii)$$

For the given equations we have:

$$\vec{a}_1 = \hat{i} - 2\hat{j} + 3\hat{k}$$

$$\vec{b}_1 = -\hat{i} + \hat{j} - 2\hat{k}$$

$\vec{a}_2 = \hat{i} - \hat{j} - \hat{k}$

$\vec{b}_2 = \hat{i} + 2\hat{j} - 2\hat{k}$

$\vec{a}_2 - \vec{a}_1 = (\hat{i} - \hat{j} - \hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k}) = \hat{j} - 4\hat{k}$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix} = (-2+4)\hat{i} - (2+2)\hat{j} + (-2-1)\hat{k} = 2\hat{i} - 4\hat{j} - 3\hat{k}$$

$\Rightarrow |\vec{b}_1 \times \vec{b}_2| = \sqrt{(2)^2 + (-4)^2 + (-3)^2} = \sqrt{4 + 16 + 9} = \sqrt{29}$

$\therefore (\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) = (2\hat{i} - 4\hat{j} - 3\hat{k}) \cdot (\hat{j} - 4\hat{k}) = -4 + 12 = 8$

Substituting all the values in equation (iii), we get

$$d = \left|\frac{8}{\sqrt{29}}\right| = \frac{8}{\sqrt{29}}$$

Thus, the shortest distance between the lines is $\frac{8}{\sqrt{29}}$ units.

Exercise 11.3

1. In each of the following cases, determine the direction cosines of the normal to the plane and the distance from the origin.

(A) $z = 2$

(B) $x + y + z = 1$

(C) $2x + 3y - z = 5$

(D) $5y + 8 = 0$

Solution:

(a) Given:

The equation of the plane is $z = 2$

The above equation can be rewritten as $0x + 0y + z = 2 \quad \ldots(i)$

The direction ratios of the normal are $0, 0$ and 1.

$\therefore \sqrt{0^2 + 0^2 + 1^2} = 1$

Dividing both sides of equation (i) by 1, we get:

$0 \times x + 0 \times y + 1 \times z = 2$

Thus, this is of the form $lx + my + nz = d$ where l, m, n are the direction cosines of normal to the plane and d is the distance of the perpendicular drawn from the origin.

Hence, the direction cosines are $0, 0$ and 1.

The distance of the plane from the origin is 2 units.

(b) Given:

The equation of the plane is $x + y + z = 1 \quad \ldots(i)$

The direction ratios of normal are $1, 1$ and 1.

$\therefore \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{3}$

Dividing both sides of equation (i) by $\sqrt{3}$, we get:

$\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z = \frac{1}{\sqrt{3}} \quad \ldots(ii)$

Thus, this equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

Hence, the direction cosines of the normal are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$ and the distance of normal from the origin is $\frac{1}{\sqrt{3}}$ units.

(c) Given:

The equation of the plane is $2x + 3y - z = 5 \quad \ldots(i)$

The direction ratios of normal are $2, 3$ and -1.

$\therefore \sqrt{(2)^2 + (3)^2 + (-1)^2} = \sqrt{14}$

Dividing both sides of equation (i) by $\sqrt{14}$, we get:

$\frac{2}{\sqrt{14}}x + \frac{3}{\sqrt{14}}y - \frac{1}{\sqrt{14}}z = \frac{5}{\sqrt{14}}$

Thus, this equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

Hence, the direction cosines of the normal to the plane are $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$ and $\frac{-1}{\sqrt{14}}$ and the distance of normal from the origin is $\frac{5}{\sqrt{14}}$ units.

(d) Given:

The equation of the plane is $5y + 8 = 0$

$\Rightarrow 0x - 5y + 0z = 8 \quad \ldots(i)$

The direction ratios of normal are $0, -5$ and 0.

$\therefore \sqrt{0 + (-5)^2 + 0} = 5$

Dividing both sides of equation (i) by 5, we get:

$-y = \frac{8}{5}$

Thus, this equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

Hence, the direction cosines of the normal to the plane are $0, -1$ and 0 and the distance of normal from the origin is $\frac{8}{5}$ units.

2. Find the vector equation of a plane which is at a distance of 7 units from the origin and normal to the vector $3\hat{i} + 5\hat{j} - 6\hat{k}$.

Solution:

Given:

The normal vector is, $\vec{n} = 3\hat{i} + 5\hat{j} - 6\hat{k}$

$$\therefore \hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{(3)^2 + (5)^2 + (6)^2}} = \frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{70}}$$

We know that the equation of the plane with position vector $\vec{r}$ given by $\vec{r}.\hat{n} = d$

$$\Rightarrow \hat{r}.\left(\frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{70}}\right) = 7$$

Hence, the vector equation of the required plane is $\hat{r}.\left(\frac{3\hat{i}+5\hat{j}-6\hat{k}}{\sqrt{70}}\right) = 7$

3. Find the Cartesian equation of the following planes:

(a) $\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 2$

(b) $\vec{r} \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) = 1$

(c) $\vec{r} \cdot [(s - 2t)\hat{i} + (3 - t)\hat{j} + (2s + t)\hat{k}] = 15$

Solution:

(a) Given:

The equation of the plane is $\vec{r}.(\hat{\imath} + \hat{\jmath} - \hat{k}) = 2$ ...(i)

For any arbitrary point $P(x, y, z)$ on the plane, the position vector $\vec{r}$ is given by,

$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Thus, by substituting the value of $\vec{r}$ in equation (i), we get:

$(x\hat{\imath} + y\hat{\jmath} + z\hat{k})(\hat{\imath} + \hat{\jmath} - \hat{k}) = 2$

$(x \times 1) + (y \times 1) + (z \times -1) = 2$

$\Rightarrow x + y - z = 2$

Therefore, $x + y - z = 2$ is the required cartesian equation of the given plane.

(b) Given:

The equation of the plane is $\vec{r}.(2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}) = 1$ ...(i)

For any arbitrary point $P(x, y, z)$ on the plane, the position vector $\vec{r}$ is given by,

$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Thus, by substituting the value of $\vec{r}$ in equation (i), we obtain

$(x\hat{\imath} + y\hat{\jmath} + z\hat{k})(2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}) = 1$

$\Rightarrow 2x + 3y - 4z = 2$

Therefore, $2x + 3y - 4z = 2$ is the required cartesian equation of the given plane.

(c) Given:

The equation of the plane is $\vec{r}.[(s - 2t)\hat{\imath} + (3 - t)\hat{\jmath} + (2s + t)\hat{k}] = 15$ ...(i)

For any arbitrary point $P(x, y, z)$ on the plane, position vector $\vec{r}$ is given by,

$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Thus, by substituting the value of $\vec{r}$ in equation (i), we obtain

$(x\hat{\imath} + y\hat{\jmath} + z\hat{k})[(s - 2t)\hat{\imath} + (3 - t)\hat{\jmath} + (2s + t)\hat{k}] = 15$

$\Rightarrow (s - 2t)x + (3 - t)y + (2s + t)z = 15$

Therefore, $(s - 2t)x + (3 - t)y + (2s + t)z = 15$ is the required cartesian equation of the given plane.

4. In the following cases, find the coordinates of the foot of the perpendicular drawn from the origin.

(A) $2x + 3y + 4z - 12 = 0$

(B) $3y + 4z - 6 = 0$

(C) $x + y + z = 1$

(D) $5y + 8 = 0$

Solution:

(A) Given:

The equation of the plane is $2x + 3y + 4z - 12 = 0$

Consider the coordinates of the foot of perpendicular P from the origin to the plane be (x_1, y_1, z_1).

$2x + 3y + 4z - 12 = 0$

$\Rightarrow 2x + 3y + 4z = 12 \quad \dots(i)$

The direction ratios of normal are 2, 3 and 4.

$\therefore \sqrt{(2)^2 + (3)^2 + (4)^2} = \sqrt{29}$

On dividing both sides of equation (i) by $\sqrt{29}$, we obtain

$\frac{2}{\sqrt{29}}x + \frac{3}{\sqrt{29}}y + \frac{4}{\sqrt{29}}z = \frac{12}{\sqrt{29}}$

Thus, this equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by (ld, md, nd).

So, $\left(\frac{2}{\sqrt{29}} \times \frac{12}{\sqrt{29}}, \frac{3}{\sqrt{29}} \times \frac{12}{\sqrt{29}}, \frac{4}{\sqrt{29}} \times \frac{12}{\sqrt{29}}\right) = \left(\frac{24}{29}, \frac{36}{49}, \frac{48}{29}\right)$.

Hence, the coordinates of the foot of the perpendicular are $\left(\frac{24}{29}, \frac{36}{49}, \frac{48}{29}\right)$.

(B) Given:

The equation of the plane is $3y + 4z - 6 = 0$

Consider the coordinates of the foot of perpendicular P from the origin to the plane be (x_1, y_1, z_1).

$3y + 4z - 6 = 0$

$\Rightarrow 0x + 3y + 4z = 6 \quad \dots(i)$

The direction ratios of the normal are 0, 3 and 4.

$\therefore \sqrt{0 + 3^2 + 4^2} = 5$

On dividing both sides of equation (i) by 5, we obtain

$0x + \frac{3}{5}y + \frac{4}{5}z = \frac{6}{5}$

Thus, this equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by (ld, md, nd).

So, $\left(0, \frac{3}{5} \times \frac{6}{5}, \frac{4}{5} \times \frac{6}{5}\right) = \left(0, \frac{18}{25}, \frac{24}{25}\right)$

Hence, the coordinates of the foot of the perpendicular are $\left(0, \frac{18}{25}, \frac{24}{25}\right)$.

(C) Given:

The equation of the plane is $x + y + z = 1$

Consider the coordinates of the foot of perpendicular P from the origin to the plane be (x_1, y_1, z_1).

$x + y + z = 1 \quad \ldots(i)$

The direction ratios of the normal are 1, 1, and 1.

$\therefore \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

On dividing both sides of equation (i) by $\sqrt{3}$, we obtain

$\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z = \frac{1}{\sqrt{3}}$

Thus, this equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by (ld, md, nd).

$\left(\frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{3}}\right) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

Hence, the coordinates of the foot of the perpendicular are $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

(D) Given:

The equation of the plane is $5y + 8 = 0$

Consider the coordinates of the foot of perpendicular P from the origin to the plane be (x_1, y_1, z_1).

$5y + 8 = 0$

$\Rightarrow 0x - 5y + 0z = 8 \quad \ldots(i)$

The direction ratios of the normal are $0, -5$, and 0.

$\therefore \sqrt{0 + (-5)^2 + 0} = 5$

On dividing both sides of equation (i) by 5, we obtain

$-y = \frac{8}{5}$

Thus, this equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by (ld, md, nd).

$\left(0, -1 \times \frac{8}{5}, 0\right) = \left(0, -\frac{8}{5}, 0\right).$

Hence, the coordinates of the foot of the perpendicular are $\left(0, -\frac{8}{5}, 0\right)$

5. Find the vector and Cartesian equation of the planes

(a) that passes through the point $(1, 0, -2)$ and the normal to the plane is $\hat{i} + \hat{j} - \hat{k}$.

(b) that passes through the point $(1, 4, 6)$ and the normal vector to the plane is $\hat{i} - 2\hat{j} + \hat{k}$.

Solution:

(a) Given:

The position vector of point $(1, 0, -2)$ is $\vec{a} = \hat{i} - 2\hat{k}$

So, the normal vector $\vec{N}$ perpendicular to the plane is $\vec{N} = \hat{i} + \hat{j} - \hat{k}$

The vector equation of the plane is given by, $(\vec{r} - \vec{a}).\vec{N} = 0$

$\Rightarrow [\vec{r} - (\hat{i} - 2\hat{k})] \cdot (\hat{i} + \hat{j} - \hat{k}) = 0 \quad \ldots(i)$

$\vec{r}$ is the position vector of any point $P(x, y, z)$ in the plane.

$\therefore \ \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Hence, equation (i) becomes

$[(x\hat{i} + y\hat{j} + z\hat{k}) - (\hat{i} - 2\hat{k})] \cdot (\hat{i} + \hat{j} - \hat{k}) = 0$

$\Rightarrow [(x - 1)\hat{i} + y\hat{j} + (z + 2)\hat{k}] \cdot (\hat{i} + \hat{j} - \hat{k}) = 0$

$\Rightarrow (x - 1) + y - (z + 2) = 0$

$\Rightarrow x + y - z - 3 = 0$

$\Rightarrow x + y - z = 3$

Therefore, $x + y - z = 3$ is the cartesian equation of the required plane.

(b) The position vector of the point $(1, 4, 6)$ is $\vec{a} = \hat{i} + 4\hat{j} + 6\hat{k}$

The normal vector $\vec{N}$ perpendicular to the plane is $\vec{N} = \hat{i} - 2\hat{j} + \hat{k}$

The vector equation of the plane is given by, $(\vec{r} - \vec{a}).\vec{N} = 0$

$\Rightarrow [\vec{r} - (\hat{i} + 4\hat{j} + 6\hat{k})] \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0 \quad \ldots(i)$

$\vec{r}$ is the position vector of any point $P(x, y, z)$ in the plane.

$\therefore \ \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Hence, equation (i) becomes

$[(x\hat{i} + y\hat{j} + z\hat{k}) - (\hat{i} + 4\hat{j} + 6\hat{k})] \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0$

$\Rightarrow [(x - 1)\hat{i} + (y - 4)\hat{j} + (z - 6)\hat{k}] \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0$

$\Rightarrow (x - 1) - 2(y - 4) + (z - 6) = 0$

$\Rightarrow x - 2y + z + 1 = 0$

Therefore, $x - 2y + z + 1 = 0$ is the Cartesian equation of the required plane.

6. Find the equations of the planes that passes through three points.

(a) $(1, 1, -1), (6, 4, -5), (-4, -2, 3)$

(b) $(1, 1, 0), (1, 2, 1), (-2, 2, -1)$

Solution:

(a) The given points are $A(1, 1, -1)$, $B(6, 4, -5)$ and $C(-4, -2, 3)$.

$$\begin{vmatrix} 1 & 1 & -1 \\ 6 & 4 & -5 \\ -4 & -2 & 3 \end{vmatrix} = (12 - 10) - (18 - 20) - (-12 + 16)$$

$= 2 + 2 - 4$

$= 0$

Since A, B, C are collinear points, there will be infinite number of planes passing through the given points.

(b) The given points are $A(1, 1, 0)$, $B(1, 2, 1)$ and $C(-2, 2, -1)$.

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -2 & 2 & -1 \end{vmatrix} = (-2-2)-(2+2) = -8 \neq 0$$

Therefore, a plane will pass through the points A, B, and C.

We know that the equation of the plane through the points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and (x_3, y_3, z_3) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-1 & z \\ 0 & 1 & 1 \\ -3 & 1 & -1 \end{vmatrix} = 0$$

$\Rightarrow (-2)(x-1) - 3(y-1) + 3z = 0$

$\Rightarrow -2x - 3y + 3z + 2 + 3 = 0$

$\Rightarrow -2x - 3y + 3z = -5$

$\Rightarrow 2x + 3y - 3z = 5$

Thus, $2x + 3y - 3z = 5$ is the Cartesian equation of the required plane.

7. Find the intercepts cut off by the plane $2x + y - z = 5$

Solution:

Given:

$2x + y - z = 5 \quad \ldots(i)$

On dividing both sides of equation (i) by 5, we obtain

$$\frac{2}{5}x + \frac{y}{5} - \frac{z}{5} = 1$$

$$\Rightarrow \frac{x}{\frac{5}{2}} + \frac{y}{5} + \frac{z}{-5} = 1 \quad \ldots(ii)$$

We know that the equation of a plane in intercept form is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, where a, b, c are the intercepts cut off by the plane at x, y and z axes respectively.

Thus, for the given equation,

$a = \frac{5}{2}, b = 5$ and $c = -5$

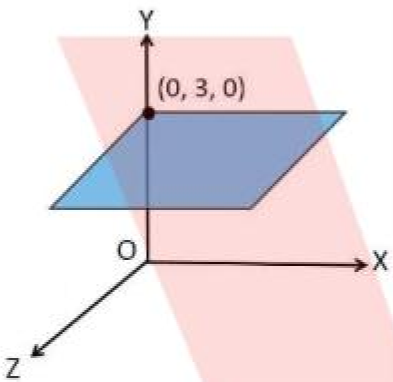
Therefore, the intercepts cut off by the plane are $\frac{5}{2}$, 5 and -5

8. Find the equation of the plane with intercept 3 on the y-axis and parallel to ZOX plane.

Solution:

Given:

The plane is parallel to ZOX plane is as shown in figure.



The equation of the plane with intercepts a, b, c on x, y, and z axis respectively.

Thus, the intercept on x-axis = 0

$\therefore a = 0$

And, the intercept on z-axis = 0

$\therefore c = 0$

Since the y-intercept of the plane is 3,

$\therefore\ b = 3$

So, equation of a plane

$\frac{x}{0} + \frac{y}{3} + \frac{z}{0} = 1$

$0 + \frac{y}{3} + 0 = 1$

$\frac{y}{3} = 1$

$y = 3$

Hence, the equation of the required plane is $y = 3$

9. Find the equation of the plane through the intersection of the planes $3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$ and the point $(2, 2, 1)$.

Solution:

The equation of any plane through the intersection of the planes,

$3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$ is

$(3x - y + 2z - 4) + \alpha(x + y + z - 2) = 0$, where $\alpha \in R$ ...(i)

The plane passes through the point $(2, 2, 1)$.

Thus, this point will satisfy equation (i)

$\therefore (3 \times 2 - 2 + 2 \times 1 - 4) + \alpha(2 + 2 + 1 - 2) = 0$

$\Rightarrow 2 + 3\alpha = 0$

$\Rightarrow \alpha = -\frac{2}{3}$

By substituting $\alpha = -\frac{2}{3}$ in equation (i), we obtain

$(3x - y + 2z - 4) - \frac{2}{3}(x + y + z - 2) = 0$

$\Rightarrow 3(3x - y + 2z - 4) - 2(x + y + z - 2) = 0$

$\Rightarrow (9x - 3y + 6z - 12) - 2(x + y + z - 2) = 0$

$\Rightarrow 7x - 5y + 4z - 8 = 0$

Therefore, $7x - 5y + 4z - 8 = 0$ is the required equation of the plane.

10. Find the vector equation of the plane passing through the intersection of the planes

$\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 7, \vec{r} \cdot (2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$ and through the point $(2, 1, 3)$.

Solution:

It is given that the plane passes through $\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 7$ and $\vec{r} \cdot (2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$

The equations of the planes are

$\Rightarrow \vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) - 7 = 0$ ...(i)

$\vec{r}\cdot(2\hat{i}+5\hat{j}+3\hat{k})-9=0$ ...(ii)

So, the equation of any plane through the intersection of the planes given in equations (i) and (ii) is given by,

$[\vec{r}\cdot(2\hat{i}+2\hat{j}-3\hat{k})-7]+\lambda[\vec{r}\cdot(2\hat{i}+5\hat{j}+3\hat{k})-9]=0$, where $\lambda\in R$

$\vec{r}\cdot[(2\hat{i}+2\hat{j}-3\hat{k})+\lambda(2\hat{i}+5\hat{j}+3\hat{k})]=9\lambda+7$

$\vec{r}\cdot[(2+2\lambda)\hat{i}+(2+5\lambda)\hat{j}+(3\lambda-3)\hat{k}]=9\lambda+7$ ...(iii)

The plane passes through the point $(2,1,3)$.

Therefore, its position vector is given by,

$\vec{r}=2\hat{i}+2\hat{j}+3\hat{k}$

By substituting in equation (iii), we obtain

$(2\hat{i}+\hat{j}-3\hat{k})\cdot[(2+2\lambda)\hat{i}+(2+5\lambda)\hat{j}+(3\lambda-3)\hat{k}]=9\lambda+7$

$\Rightarrow(2+2\lambda)+(2+5\lambda)+(3\lambda-3)=9\lambda+7$

$\Rightarrow 18\lambda-3=9\lambda+7$

$\Rightarrow 9\lambda=10$

$\Rightarrow \lambda=\frac{10}{9}$

Hence, by substituting $\lambda=\frac{10}{9}$ in equation (iii), we obtain

$\vec{r}\cdot\left(\frac{38}{9}\hat{i}+\frac{68}{9}\hat{j}+\frac{3}{9}\hat{k}\right)=17$

$\Rightarrow\vec{r}\cdot(38\hat{i}+68\hat{j}+3\hat{k})=153$

Therefore, $\vec{r}\cdot(38\hat{i}+68\hat{j}+3\hat{k})=153$ is the vector equation of the required plane.

11. Find the equation of the plane through the line of intersection of the planes $x+y+z=1$ and $2x+3y+4z=5$ which is perpendicular to the plane $x-y+z=0$

Solution:

The equation of the plane through the intersection of the planes, $x+y+z=1$ and $2x+3y+4z=5$ is

$(x+y+z-1)+\lambda(2x+3y+4z-5)=0$

$\Rightarrow (2\lambda + 1)x + (3\lambda + 1)y + (4\lambda + 1)z - (5\lambda + 1) = 0 \quad \ldots(i)$

The direction ratios, a_1, b_1, c_1 of this plane area $(2\lambda + 1), (3\lambda + 1)$ and $(4\lambda + 1)$.

The plane in equation (i) is perpendicular to $x - y + z = 0$

Its direction ratios, a_2, b_2, c_2 are $1, -1$ and 1.

Since the planes are perpendicular,

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\Rightarrow (2\lambda + 1) - (3\lambda + 1) + (4\lambda + 1) = 0$

$\Rightarrow 3\lambda + 1 = 0$

$\Rightarrow \lambda = -\frac{1}{3}$

By substituting $\lambda = -\frac{1}{3}$ in equation (i), we obtain

$\frac{1}{3}x - \frac{1}{3}z + \frac{2}{3} = 0$

$\Rightarrow x - z + 2 = 0$

Therefore, $x - z + 2 = 0$ is the required equation of the plane.

12. Find the angle between the planes whose vector equations are

$\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 5$ and $\vec{r} \cdot (3\hat{i} - 3\hat{j} + 5\hat{k}) = 3$

Solution:

The equations of the given planes are $\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 5$ and $\vec{r} \cdot (3\hat{i} - 3\hat{j} + 5\hat{k}) = 3$

We know that if $\vec{n}_1$ and $\vec{n}_2$ are $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ normal to the planes, then the angle between them is given by Q,

$\cos Q = \left|\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right| \quad \ldots(i)$

Here, $\vec{n}_1 = 2\hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{n}_2 = 3\hat{i} - 3\hat{j} + 5\hat{k}$

$\therefore \vec{n}_1 \cdot \vec{n}_2 = \left(2\hat{i} + 2\hat{j} - 3\hat{k}\right)\left(3\hat{i} - 3\hat{j} + 5\hat{k}\right)$

$= 2.3 + 2 \cdot (-3) + (-3) \cdot 5 = -15$

$|\vec{n}_1| = \sqrt{(2)^2 + (2)^2 + (-3)^2} = \sqrt{17}$

$|\vec{n}_2| = \sqrt{(3)^2 + (-3)^2 + (5)^2} = \sqrt{43}$

By substituting the value of $\vec{n} \cdot \vec{n}_2, |\vec{n}_1|$ and $|\vec{n}_2|$

$$\cos Q = \left|\frac{-15}{\sqrt{17}.\sqrt{43}}\right|$$

$$\Rightarrow \cos Q = \frac{15}{\sqrt{731}}$$

$$\Rightarrow Q = \cos^{-1}\left(\frac{15}{\sqrt{731}}\right)$$

Hence, the angle between the planes for the given vector equations is $\cos^{-1}\left(\frac{15}{\sqrt{731}}\right)$.

13. In the following cases, determine whether the given planes are parallel or perpendicular and in case they are neither, find the angles between them.

(a) $7x + 5y + 6z + 30 = 0$ and $3x - y - 10z + 4 = 0$

(b) $2x + y + 3z - 2 = 0$ and $x - 2y + 5 = 0$

(c) $2x - 2y + 4z + 5 = 0$ and $3x - 3y + 6z - 1 = 0$

(d) $2x - y + 3z - 1 = 0$ and $2x - y + 3z + 3 = 0$

(e) $4x + 8y + z - 8 = 0$ and $y + z - 4 = 0$

Solution:

The direction ratios of normal to the plane, $L_1: a_1x + b_1y + c_1z = 0$ are a_1, b_1, c_1 and $L_2: a_1x + b_2y + c_2z = 0$ are a_2, b_2, c_2

$L_1 \| L_2$, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$L_1 \perp L_2$, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Thus, the angle between L_1 and L_2 is given by,

$$Q = \cos^{-1}\left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2 \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}}}\right|$$

(a) The given equations of the planes are $7x + 5y + 6z + 30 = 0$ and $3x - y - 10z + 4 = 0$

Here, $a_1 = 7, b_1 = 5, c_1 = 6$

$a_2 = 3, b_2 = -1, c_2 = -10$

$a_1a_2 + b_1b_2 + c_1c_2 = 7 \times 3 + 5 \times (-1) + 6 \times (-10) = -44 \neq 0$

Hence, the given planes are not perpendicular.

$\frac{a_1}{a_2} = \frac{7}{3}, \frac{b_1}{b_2} = \frac{5}{-1} = -5, \frac{c_1}{c_2} = \frac{6}{-10} = \frac{-3}{5}$

It can be observed that, $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

Therefore, the given planes are not parallel.

So, the angle between them is given by,

$$Q = \cos^{-1}\left|\frac{7 \times 3 + 5 \times (-1) + 6 \times (-10)}{\sqrt{(7)^2 + (5)^2 + (6)^2} \times \sqrt{(3)^2 + (-1)^2 + (-10)^2}}\right|$$

$$= \cos^{-1}\left|\frac{21 - 5 - 60}{\sqrt{110} \times \sqrt{110}}\right|$$

$$= \cos^{-1}\frac{44}{110}$$

$$Q = \cos^{-1}\frac{2}{5}$$

(b) The given equations of the planes are $2x + y + 3z - 2 = 0$ and $x - 2y + 5 = 0$

Here, $a_1 = 2, b_1 = 1, c_1 = 3$ and $a_2 = 1, b_2 = -2, c_2 = 0$

$\therefore a_1a_2 + b_1b_2 + c_1c_2 = 2 \times 1 + 1 \times (-2) + 3 \times 0 = 0$

Therefore, the given planes are perpendicular to each other.

(c) The given equations of the given planes are $2x - 2y + 4z + 5 = 0$ and $3x - 3y + 6z - 1 = 0$

Here, $a_1 = 2, b_1 - 2, c_1 = 4$ and $a_2 = 3, b_2 = -3, c_2 = 6$

$a_1a_2 + b_1b_2 + c_1c_2 = 2 \times 3 + (-2)(-3) + 4 \times 6 = 6 + 6 + 24 = 36 \neq 0$

Hence, the given planes are not perpendicular to each other.

$\frac{a_1}{a_2} = \frac{2}{3}, \frac{b_1}{b_2} = \frac{-2}{-3} = \frac{2}{3}$ and $\frac{c_1}{c_2} = \frac{4}{6} = \frac{2}{3}$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Therefore, the given planes are parallel to each other.

(d) The given equations of the planes are $2x - y + 3z - 1 = 0$ and $2x - y + 3z + 3 = 0$

$a_1 = 2, b_1 = -1, c_1 = 3$ and $a_2 = 2, b_2 = -1, c_2 = 3$

Here,

$\frac{a_1}{a_2} = \frac{2}{2} = 1, \frac{b_1}{b_2} = \frac{-1}{-1} = 1$ and $\frac{c_1}{c_2} = \frac{3}{3} = 1$

$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Therefore, the given lines are parallel to each other.

(e) The given equations of the given planes are $4x + 8y + z - 8 = 0$ and $y + z - 4 = 0$

$a_1 = 4, b_1 = 8, c_1 = 1$ and $a_2 = 0, b_2 = 1, c_2 = 1$

Here,

$a_1a_2 + b_1b_2 + c_1c_2 = 4 \times 0 + 8 \times 1 + 1 = 9 \neq 0$

Hence, the given lines are not perpendicular to each other.

$\frac{a_1}{a_2} = \frac{4}{0}, \frac{b_1}{b_2} = \frac{8}{1} = 8, \frac{c_1}{c_2} = \frac{1}{1} = 1$

$\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

Thus, the given lines are not parallel to each other.

So, the angle between the planes is given by,

$$Q = \cos^{-1}\left|\frac{4 \times 0 + 8 \times 1 + 1 \times 1}{\sqrt{4^2 + 8^2 + 1^2} \times \sqrt{0^2 + 1^2 + 1^2}}\right|$$

$$= \cos^{-1}\left|\frac{9}{9 \times \sqrt{2}}\right|$$

$$= \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^o$$

14. In the following cases, find the distance of each of the given points from the corresponding given plane.

Point Plane

(a) $(0, 0, 0)$ $3x - 4y + 12z = 3$

(b) $(3, -2, 1)$ $2x - y + 2z + 3 = 0$

(c) $(2, 3, -5)$ $x + 2y - 2z = 9$

(d) $(-6,0,0)$ $2x-3y+6z-2=0$

Solution:

We know that the distance between a point $p(x_1, y_1, z_1)$ and a plane $Ax+By+Cz=D$ is given by,

$d=\left|\frac{Ax_1+By_1+Cz_1-D}{\sqrt{A^2+B^2+C^2}}\right|$ ...(i)

(a) The given point is $(0,0,0)$ and the plane is $3x-4y+12z=3$

$\therefore d=\left|\frac{3\times0-4\times0+12\times0-3}{\sqrt{(3)^2+(-4)^2+(12)^2}}\right|=\frac{3}{\sqrt{169}}=\frac{3}{13}$

(b) The given point is $(3,-2,1)$ and the plane is $2x-y+2z+3=0$

$\therefore d=\left|\frac{2\times3-(-2)+2\times1+3}{\sqrt{(2)^2+(-1)^2+(2)^2}}\right|=\left|\frac{13}{3}\right|=\frac{13}{3}$

(c) The given point is $(2,3,-5)$ and the plane is $x+2y-2z=9$

$\therefore d=\left|\frac{2+2\times3-2(-5)-9}{\sqrt{(1)^2+(2)^2+(-2)^2}}\right|=\frac{9}{3}=3$

(d) The given point is $(-6,0,0)$ and the plane is $2x-3y+6z-2=0$

$\therefore d=\left|\frac{2(-6)-3\times0+6\times0-2}{\sqrt{(2)^2+(-3)^2+(6)^2}}\right|=\left|\frac{-14}{\sqrt{49}}\right|=\frac{14}{7}=2$

Miscellaneous

1. Show that the line joining the origin to the point $(2,1,1)$ is perpendicular to the line determined by the points $(3,5,-1),(4,3,-1)$.

Solution:

Let OA be the line joining the origin, $O(0,0,0)$ and the point, $A(2,1,1)$.

Also, let BC be the line joining the points, $B(3,5,-1)$ and $C(4,3,-1)$.

The direction ratios of OA are $2,1$ and 1 and the direction ratios of BC are $(4-3)=1,(3-5)=-2$ and $(-1+1)=0$

Now,

$a_1a_2 + b_1b_2 + c_1c_2 = 2 \times 1 + 1(-2) + 1 \times 0 = 2 - 2 = 0$

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

Therefore, OA is perpendicular to BC.

2. If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines, show that the direction cosines of the line perpendicular to both of these are $m_1n_2 - m_2n_1, n_1l_2 - n_2l_1, l_1m_2 - l_2m_1$.

Solution:

It is given that l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines. Thus,

$l_1l_2 + m_1m_2 + n_1n_2 = 0 \quad \ldots(i)$

$l_1^2 + m_1^2 + n_1^2 = 1 \quad \ldots(ii)$

$l_2^2 + m_2^2 + n_2^2 = 1 \quad \ldots(iii)$

Let l, m, n be the direction cosines of the line which is perpendicular to the line with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2.

$\therefore ll_1 + mm_1 + nn_1 = 0$

$ll_2 + mm_2 + nn_2 = 0$

$$\therefore \frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1}$$

$$\Rightarrow \frac{l^2}{(m_1n_2 - m_2n_1)^2} = \frac{m^2}{(n_1l_2 - n_2l_1)^2} = \frac{n^2}{(l_1m_2 - l_2m_1)^2}$$

$$\Rightarrow \frac{l^2 + m^2 + n^2}{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2} \quad \ldots(iv)$$

l, m, n are the direction cosines of the line.

$\therefore l^2 + m^2 + n^2 = 1 \quad \ldots(v)$

We know that,

$(l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1l_2 + m_1m_2 + n_1n_2)^2$

$= (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2$

From (i), (ii) and (iii), we obtain

$\Rightarrow 1 \times 1 - 0 = (m_1n_2 + m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2$

$\therefore\ (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2 = 1 \quad \ldots(vi)$

Substituting the values from equations (v) and (vi) in equation (iv), we obtain

$$\frac{l^2}{(m_1n_2 - m_2n_1)^2} = \frac{m^2}{(n_2l_2 - n_2l_1)^2} = \frac{n^2}{(l_1m_2 - l_2m_1)^2} = 1$$

$\Rightarrow l = m_1n_2 - m_2n_1, m = n_1l_2 - n_2l_1, n = l_1m_2 - l_2m_1$

Hence, the direction cosines of the required line are $m_1n_2 - m_2n_1, n_1l_2 - n_2l_1$ and $l_1m_2 - l_2m_1$.

3. Find the angle between the lines whose direction ratios are a, b, c and $b - c, c - a, a - b$.

Solution:

The angle Q between the lines with direction cosines, a, b, c and $b - c, c - a, a - b$ is given by,

$$\cos Q = \left|\frac{a(b-c) + b(c-a) + c(a-b)}{\sqrt{a^2 + b^2 + c^2} + \sqrt{(b-c)^2 + (c-a)^2 + (a-b)^2}}\right|$$

$\Rightarrow \cos Q = 0$

$\Rightarrow Q = \cos^{-1} 0$

$\Rightarrow Q = 90^o$

Hence, the angle between the lines is 90^o.

4. Find the equation of a line parallel to x-axis and passing through the origin.

Solution:

The line parallel to x-axis and passing through the origin is x-axis itself.

Let A be a point on x-axis.

Thus, the coordinates of A are given by $(a, 0, 0)$, where $a \in R$.

Direction ratios of OA are $(a - 0) = a, 0, 0$

The equation of OA is given by,

$\frac{x-0}{a} = \frac{y-0}{0} = \frac{z-0}{0}$

$\Rightarrow \frac{x}{1} = \frac{y}{0} = \frac{z}{0} = a$

Therefore, the equation of line parallel to x-axis and passing through origin is

$\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$

5. If the coordinates of the points A, B, C, D be $(1, 2, 3), (4, 5, 7), (-4, 3, -6)$ and $(2, 9, 2)$ respectively, then find the angle between the lines AB and CD.

Solution:

It is given that the coordinates of A, B, C and D are $(1, 2, 3), (4, 5, 7), (-4, 3, -6)$ and $(2, 9, 2)$ respectively.

The direction ratios of AB are $(4 - 1) = 3, (5 - 2) = 3$ and $(7 - 3) = 4$

The direction ratios of CD are $\left(2 - (-4)\right) = 6, (9 - 3) = 6$ and $\left(2 - (-6)\right) = 8$

It can be seen that $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{1}{2}$

Therefore, AB is parallel to CD.

Hence, the angle between AB and CD is either 0^o or 180^o.

6. If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ are perpendicular, find the value of k.

Solution:

Given: Two lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ are perpendicular to $a_1a_2 + b_1b_2 + c_1c_2 = 0$

The direction of ratios of the lines, $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ are $-3, 2k, 2$ and $3k, 1, -5$ respectively.

We know that, two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2 are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\therefore -3(3k) + 2k \times 1 + 2(-5) = 0$

$\Rightarrow -9k + 2k - 10 = 0$

$\Rightarrow 7k = -10$

$\Rightarrow k = \frac{-10}{7}$

Therefore, for $k = -\frac{10}{7}$, the given lines are perpendicular to each other.

7. Find the vector equation of the line passing through $(1, 2, 3)$ and perpendicular to the plane $\vec{r} \cdot (\hat{i} + 2\hat{j} - 5\hat{k}) + 9 = 0$

Solution:

Given: The position vector of the point $(1, 2, 3)$ is $\vec{r}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$

The direction ratios of the normal to the $\vec{r} \cdot (\hat{i} + 2\hat{j} - 5\hat{k}) + 9 = 0$ plane are $1, 2$ and the normal vector is $\vec{N} = \hat{i} + 2\hat{j} - 5\hat{k}$

The equation of a line passing through a point and perpendicular to the given plane is given by $\vec{l} = \vec{r} + \lambda\vec{N}, \quad \lambda \in R$

$\Rightarrow \vec{l} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} - 5\hat{k})$

Therefore, the required equation is $\vec{l} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} - 5\hat{k})$.

8. Find the equation of the plane passing through (a, b, c) and parallel to the plane

$\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$

Solution:

It is given that equation parallel to the plane is $\vec{r}_1 \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$

$\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = \lambda \quad \ldots(i)$

The plane passes through the point (a, b, c).

Therefore, the position vector $\vec{r}$ of this point is $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$

Thus, equation (i) becomes

$(a\hat{i} + b\hat{j} + c\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) = \lambda$

$\Rightarrow a + b + c = \lambda$

Substituting $\lambda = a + b + c$ in equation (i), we obtain

$\vec{r}.(\hat{i} + \hat{j} + \hat{k}) = a + b + c \quad \ldots$(ii)

This is the vector equation of the required plane.

Substituting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in equation (ii), we obtain

$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) = a + b + c$

$\Rightarrow x + y + z = a + b + c$

Hence, the equation of the plane in cartesian form is $x + y + z = a + b + c$

9. Find the shortest distance between lines $\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$ and $\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$

Solution:

Given lines are:

$\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k}) \quad \ldots$(i)

$\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k}) \quad \ldots$(ii)

We know that, the shortest distance between two lines, $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ is given by

$d = \left|\frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right| \quad \ldots$(iii)

Comparing $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ to equations (i) and (ii), we obtain

$\vec{a}_1 = 6\hat{i} + 2\hat{j} + 2\hat{k}$

$\vec{b}_1 = \hat{i} - 2\hat{j} + 2\hat{k}$

$\vec{a}_2 = -4\hat{i} - \hat{k}$

$\vec{b}_2 = 3\hat{i} - 2\hat{j} - 2\hat{k}$

$\Rightarrow \vec{a}_2 - \vec{a}_1 = (-4\hat{i} - \hat{k}) - (6\hat{i} + 2\hat{j} + 2\hat{k}) = -10\hat{i} - 2\hat{j} - 3\hat{k}$

$$\Rightarrow \vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix}$$

$= (4 + 4)\hat{i} - (-2 - 6)\hat{j} + (-2 + 6)\hat{k} = 8\hat{i} + 8\hat{j} + 4\hat{k}$

$\therefore |\vec{b}_1 \times \vec{b}_2| = \sqrt{(8)^2 + (8)^2 + (4)^2} = 12$

$(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) = (8\hat{i} + 8\hat{j} + 4\hat{k}) \cdot (-10\hat{i} - 2\hat{j} - 3\hat{k})$

$= -80 - 16 - 12 = -108$

Substituting all the values in equation (1), we obtain

$$d = \left|\frac{-108}{12}\right| = 9$$

Hence, the shortest distance between the two given lines is 9 units.

10. Find the coordinates of the point where the line through $(5, 1, 6)$ and $(3, 4, 1)$ crosses the YZ-plane.

Solution:

It is given that the line passes through $(5, 1, 6)$ and $(3, 4, 1)$ crosses the YZ-plane

We know that, the equation of the line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2)

is $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

The line passing through the points $(5, 1, 6)$ and $(3, 4, 1)$ is given by

$$\frac{x-5}{3-5} = \frac{y-1}{4-1} = \frac{z-6}{1-6}$$

$\Rightarrow \frac{x-5}{-2} = \frac{y-1}{3} = \frac{z-6}{-5} = k$

$\Rightarrow x = 5 - 2k, y = 3k + 1, z = 6 - 5k$

Any point on the line is of the form $(5 - 2k, 3k + 1, 6 - 5k)$.

The equation of YZ-plane is $x = 0$

As the line passes through YZ-plane, we get

$5 - 2k = 0$

$\Rightarrow k = \frac{5}{2}$

$\Rightarrow 3k + 1 = 3 \times \frac{5}{2} + 1 = \frac{17}{2}$

$6 - 5k = 6 - 5 \times \frac{5}{2} = \frac{-13}{2}$

Therefore, the required point is $\left(0, \frac{17}{2}, \frac{-13}{2}\right)$.

11. Find the coordinates of the point where the line through $(5, 1, 6)$ and $(3, 4, 1)$ crosses the ZX − plane.

Solution:

It is given that the line passes through $(5, 1, 6)$ and $(3, 4, 1)$ crosses the ZX-plane

We know that, the equation of the line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2)

is $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

The line passing through the points $(5, 1, 6)$ and $(3, 4, 1)$ is given by

$\frac{x-5}{3-5} = \frac{y-1}{4-1} = \frac{z-6}{1-6}$

$\Rightarrow \frac{x-5}{-2} = \frac{y-1}{3} = \frac{z-6}{-5} = k$

$\Rightarrow x = 5 - 2k, y = 3k + 1, z = 6 - 5k$

Any point on the line is of the form $(5 - 2k, 3k + 1, 6 - 5k)$.

As the line passes through ZX-plane, we get

$3k + 1 = 0$

$\Rightarrow k = -\frac{1}{3}$

Now,

$\Rightarrow x = 5 - 2k = 5 - 2\left(-\frac{1}{3}\right) = \frac{17}{3}$

$\Rightarrow z = 6 - 5k = 6 - 5\left(-\frac{1}{3}\right) = \frac{23}{3}$

Hence, the required point is $\left(\frac{17}{3}, 0, \frac{23}{3}\right)$

12. Find the coordinates of the point where the line through $(3, -4, -5)$ and $(2, -3, 1)$ crosses the plane $2x + y + z = 7$.

Solution:

The given line passes through $(3, -4, -5)$ and $(2, -3, 1)$ crosses the plane $2x + y + z = 7$.

We know that the equation of the line through the points, (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

As the line passing through the points $(3, -4, -5)$ and $(2, -3, 1)$, its equation is given by

$\frac{x-3}{2-3} = \frac{y+4}{-3+4} = \frac{z+5}{1+5}$

$\Rightarrow \frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = k$

$\Rightarrow x = 3 - k, y = k - 4, z = 6k - 5$

Thus, any point on the line is of the form $(3 - k, k - 4, 6k - 5)$.

This point lies on the plane, $2x + y + z = 7$

$\therefore 2(3 - k) + (k - 4) + (6k - 5) = 7$

$\Rightarrow 5k - 3 = 7$

$\Rightarrow k = 2$

Hence, the coordinates of the required point are $(3 - 2, 2 - 4, 6 \times 2 - 5)$ i.e., $(1, -2, 7)$.

13. Find the equation of the plane passing through the point $(-1, 3, 2)$ and perpendicular to each of the planes $x + 2y + 3z = 5$ and $3x + 3y + z = 0$.

Solution:

The equation of the plane passing through the point $(-1, 3, 2)$ is $(x+1) + b(y-3) + c(z-2) = 0$ ...(i) where, a, b, c are the direction ratios of normal to the plane.

We know that, two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are perpendicular, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

So, equation (i) is perpendicular to the plane $x + 2y + 3z = 5$

$\therefore a \times 1 + b \times 2 + c \times 3 = 0$

$\Rightarrow a + 2b + 3c = 0$ ...(ii)

Also, equation (i) is perpendicular to the plane, $3x + 3y + z = 0$

$\therefore a \times 3 + b \times 3 + c \times 1 = 0$

$\Rightarrow 3a + 3b + c = 0$ ...(iii)

From equations (ii) and (iii), we get

$$\frac{a}{2 \times 1 - 3 \times 3} = \frac{b}{3 \times 3 - 1 \times 1} = \frac{c}{1 \times 3 - 2 \times 3}$$

$\Rightarrow \frac{a}{-7} = \frac{b}{8} = \frac{c}{-3} = k(\text{say})$

$\Rightarrow a = -7k, b = 8k, c = -3k$

Substituting the values of a, b, and c in equation (i), we get

$-7k(x+1) + 8k(y-3) - 3k(z-2) = 0$

$\Rightarrow (-7x - 7) + (8y - 24) - 3z + 6 = 0$

$\Rightarrow -7x + 8y - 3z - 25 = 0$

$\Rightarrow 7x - 8y + 3z + 25 = 0$

Therefore, $7x - 8y + 3z + 25 = 0$ is the required equation of the plane.

14. If the points $(1, 1, p)$ and $(-3, 0, 1)$ be equidistant from the plane $\vec{r} \cdot (3\hat{i} + 4\hat{j} - 12\hat{k}) + 13 = 0$, then find the value of p.

Solution:

The given points are $(1, 1, p)$ and $(-3, 0, 1)$

The position vector through the point $(1, 1, p)$ is $\vec{a}_1 = \hat{\imath} + \hat{\jmath} + p\hat{k}$

Similarly, the position vector through the point $(-3, 0, 1)$ is

$\vec{a}_2 = -4\hat{\imath} + \hat{k}$

The equation of the given plane is $\vec{r} \cdot (3\hat{\imath} + 4\hat{\jmath} - 12\hat{k}) + 13 = 0$

We know that, the perpendicular distance between a point whose position vector is $\vec{a}$ and the plane, $\vec{r} \cdot \vec{N} = d$, is given by $D = \frac{|\vec{a} \cdot \vec{N} - d|}{|\vec{N}|}$

In this condition, $\vec{N} = 3\hat{\imath} + 4\hat{\jmath} - 12\hat{k}$ and $d = -13$

Thus, the distance between the point $(1, 1, p)$ and the given plane is

$$D_1 = \frac{|(\hat{\imath} + \hat{\jmath} + p\hat{k}) \times (3\hat{\imath} + 4\hat{\jmath} - 12\hat{k}) + 13|}{|3\hat{\imath} + 4\hat{\jmath} - 12\hat{k}|}$$

$$\Rightarrow D_1 = \frac{|3 + 4 - 12p + 13|}{\sqrt{3^2 + 4^2 + (-12)^2}}$$

$\Rightarrow D_1 = \frac{|20 - 12p|}{13}$ ...(i)

Similarly, the distance between the point $(-3, 0, 1)$ and the given plane is

$$D_2 = \frac{|(-3\hat{\imath} + \hat{k}) \cdot (3\hat{\imath} + 4\hat{\jmath} - 12\hat{k}) + 13|}{|3\hat{\imath} + 4\hat{\jmath} - 12\hat{k}|}$$

$$\Rightarrow D_2 = \frac{|-9 - 12 + 13|}{\sqrt{3^2 + 4^2 + (-12)^2}}$$

$\Rightarrow D_2 = \frac{8}{13}$ ...(ii)

Since, it is given that the distance between the required plane and the points, $(1, 1, p)$ and $(-3, 0, 1)$ is equal.

$\therefore D_1 = D_2$

$$\Rightarrow \frac{|20 - 12p|}{13} = \frac{8}{13}$$

$\Rightarrow 20 - 12p = 8$ or $-(20 - 12p) = 8$

$\Rightarrow 12p = 12$ or $12p = 28$

$\Rightarrow p = 1$ or $p = \frac{7}{3}$

Hence, the value of p is 1 or $\frac{7}{3}$.

15. Find the equation of the plane passing through the line of intersection of the planes $\vec{r} \cdot (\hat{\imath} + \hat{\jmath} + \hat{k}) = 1$ and $\vec{r} \cdot (2\hat{\imath} + 3\hat{\jmath} - \hat{k}) + 4 = 0$ and parallel to x-axis.

Solution:

Given planes are $\vec{r} \cdot (\hat{\imath} + \hat{\jmath} + \hat{k}) = 1$ and $\vec{r} \cdot (2\hat{\imath} + 3\hat{\jmath} - \hat{k}) + 4 = 0$

Now, $\vec{r} \cdot (\hat{\imath} + \hat{\jmath} + \hat{k}) = 1$

$\Rightarrow \vec{r} \cdot (\hat{\imath} + \hat{\jmath} + \hat{k}) - 1 = 0$

$\vec{r} \cdot (2\hat{\imath} + 3\hat{\jmath} - \hat{k}) + 4 = 0$

The equation of any plane passing through the line of intersection of these planes is

$[\vec{r} \cdot (\hat{\imath} + \hat{\jmath} + \hat{k}) - 1] + \lambda[\vec{r} \cdot (2\hat{\imath} + 3\hat{\jmath} - \hat{k}) + 4] = 0$

$\vec{r} \cdot [(2\lambda + 1)\hat{\imath} + (3\lambda + 1)\hat{\jmath} + (1 - \lambda)\hat{k}] + (4\lambda + 1) = 0$ ...(i)

Its direction ratios are $(2\lambda + 1), (3\lambda + 1)$ and $(1 - \lambda)$.

The required plane is parallel to x-axis.

Therefore, its normal is perpendicular to x-axis.

The direction ratios of x-axis are 1, 0 and 0.

$\therefore 1 \times (2\lambda + 1) + 0(3\lambda + 1) + 0(1 - \lambda) = 0$

$\Rightarrow 2\lambda + 1 = 0$

$\Rightarrow \lambda = -\frac{1}{2}$

Substituting $\lambda = -\frac{1}{2}$ in equation (i), we get

$\Rightarrow \vec{r} \cdot \left[-\frac{1}{2}\hat{\jmath} + \frac{3}{2}\hat{k}\right] + (-3) = 0$

$\Rightarrow \vec{r}(\hat{\jmath} - 3\hat{k}) + 6 = 0$

Thus, its cartesian equation is $y - 3z + 6 = 0$

Therefore, $y - 3z + 6 = 0$ is the equation of the required plane.

16. If O be the origin and the coordinates of P be $(1, 2, -3)$, then find the equation of the plane passing through P and perpendicular to OP.

Solution:

Given:

The coordinates of the points, O and P are $(0, 0, 0)$ and $(1, 2, -3)$ respectively.

Thus, the direction ratios of OP are $(1 - 0) = 1, (2 - 0) = 2$ and $(-3 - 0) = -3$

We know that, the equation of the plane passing through the point $(x_1, y_1\ z_1)$ is

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

where a, b and c are the direction ratios of normal.

Here, the direction ratios of normal are $1, 2$ and -3 and the point P is $(1, 2, -3)$.

Therefore, the equation of the required plane is

$1(x - 1) + 2(y - 2) - 3(z + 3) = 0$

$\Rightarrow x + 2y - 3z - 14 = 0$

Hence, $x + 2y - 3z - 14 = 0$ is the required equation of the plane.

17. Find the equation of the plane which contains the line of intersection of the planes $\vec{r} \cdot (\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) - 4 = 0$, $\vec{r} \cdot (2\hat{\imath} + \hat{\jmath} - \hat{k}) + 5 = 0$ and which is perpendicular to the plane

$\vec{r} \cdot (5\hat{\imath} + 3\hat{\jmath} - 6\hat{k}) + 8 = 0$

Solution:

The given equations of the planes are

$\vec{r} \cdot (\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) - 4 = 0$ ...(i)

$\vec{r} \cdot (2\hat{\imath} + \hat{\jmath} - \hat{k}) + 5 = 0$ ...(ii)

The equation of the plane passing through the line intersection of the plane given in equation (i) and equation (ii) is

$[\vec{r} \cdot (\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) - 4] + \lambda[\vec{r} \cdot (2\hat{\imath} + \hat{\jmath} - \hat{k}) + 5] = 0$

$\vec{r} \cdot [(2\lambda + 1)\hat{\imath} + (\lambda + 2)\hat{\jmath} + (3 - \lambda)\hat{k}] + (5\lambda - 4) = 0$ ...(iii)

The plane in equation (iii) is perpendicular to the plane, $\vec{r}\cdot\left(5\hat{\imath}+3\hat{\jmath}-6\hat{k}\right)+8=0$

$\therefore 5(2\lambda+1)+3(\lambda+2)-6(3-\lambda)=0$

$\Rightarrow 19\lambda-7=0$

$\Rightarrow \lambda=\frac{7}{19}$

Substituting $\lambda=\frac{7}{19}$ in equation (iii), we get

$\Rightarrow \vec{r}\cdot\left[\frac{33}{19}\hat{\imath}+\frac{45}{19}\hat{\jmath}+\frac{50}{19}\hat{k}\right]\frac{-41}{19}=0$

$\Rightarrow \vec{r}\cdot\left(33\hat{\imath}+45\hat{\jmath}+50\hat{k}\right)-41=0$ ...(iv)

This is the vector equation of the required plane.

The Cartesian equation of this plane can be obtained by substituting $\vec{r}=x\hat{\imath}+\hat{y}\hat{\jmath}+z\hat{k}$ in equation (iii).

$\left(x\hat{\imath}+y\hat{\jmath}+z\hat{k}\right)\cdot\left(33\hat{\imath}+45\hat{\jmath}+50\hat{k}\right)-41=0$

$\Rightarrow 33x+45y+50z-41=0$

Therefore, $33x+45y+50z-41=0$ is the required equation of the plane.

18. Find the distance of the point $(-1,-5,-10)$ from the point of intersection of the line $\vec{r}=2\hat{\imath}-\hat{\jmath}+2\hat{k}+\lambda\left(3\hat{\imath}+4\hat{\jmath}+2\hat{k}\right)$ and the plane $\vec{r}\cdot\left(\hat{\imath}-\hat{\jmath}+\hat{k}\right)=5$.

Solution:

The equation of the given line is

$\vec{r}=2\hat{\imath}-\hat{\jmath}+2\hat{k}+\lambda\left(3\hat{\imath}+4\hat{\jmath}+2\hat{k}\right)$ ...(i)

The equation of the given plane is

$\vec{r}\cdot\left(\hat{\imath}-\hat{\jmath}+\hat{k}\right)=5$ ...(ii)

Substituting the value of $\vec{r}$ from equation (i) in equation (ii), we get

$\left[2\hat{\imath}-\hat{\jmath}+2\hat{k}+\lambda\left(3\hat{\imath}+4\hat{\jmath}+2\hat{k}\right)\right]\cdot\left(\hat{\imath}-\hat{\jmath}+\hat{k}\right)=5$

$\Rightarrow \left[(3\lambda+2)\hat{\imath}+(4\lambda-1)\hat{\jmath}+(2\lambda+2)\hat{k}\right]\cdot\left(\hat{\imath}-\hat{\jmath}+\hat{k}\right)=5$

$\Rightarrow (3\lambda+2)-(4\lambda-1)+(2\lambda+2)=5$

$\Rightarrow \lambda = 0$

Substituting this value in equation (i), we get the equation of the line as

$\vec{r} = 2\hat{\imath} - \hat{\jmath} + 2\hat{k}$

This means that the position vector of the point of intersection of the line and the plane is $\vec{r} = 2\hat{\imath} - \hat{\jmath} + 2\hat{k}$

This shows that the point of intersection of the given line and plane is given by the coordinate, $(2, -1, 2)$.

The distance d between the points $(2, -1, 2)$ and $(-1, -5, -10)$, is

$d = \sqrt{(-1-2)^2 + (-5+1)^2 + (-10-2)^2}$

$= \sqrt{9 + 16 + 144} = \sqrt{169} = 13$

Therefore, the distance of the given point is 13.

19. Find the vector equation of the line passing through $(1, 2, 3)$ and parallel to the planes $\vec{r} = (\hat{\imath} - \hat{\jmath} + 2\hat{k}) = 5$ and $\vec{r} \cdot (3\hat{\imath} + \hat{\jmath} + \hat{k}) = 6$

Solution:

The given equation of the planes are $\vec{r} = (\hat{\imath} - \hat{\jmath} + 2\hat{k}) = 5$ and $\vec{r} \cdot (3\hat{\imath} + \hat{\jmath} + \hat{k}) = 6$

Let the required line be parallel to vector $\vec{b}$ given by,

$\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

The position vector of the point $(1, 2, 3)$ is $\vec{a} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

The equation of line passing through $(1, 2, 3)$ and parallel to $\vec{b}$ is given by,

$\vec{r} = \vec{a} + \lambda\vec{b}$

$\Rightarrow \vec{r}(\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) + \lambda(b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k})$ ...(i)

The equations of the given planes are

$\vec{r} \cdot (\hat{\imath} - \hat{\jmath} + 2\hat{k}) = 5$ ...(ii)

$\vec{r} \cdot (3\hat{\imath} + \hat{\jmath} + \hat{k}) = 6$ ...(iii)

The line in equation (i) and plane in equation (ii) are parallel.

Therefore, the normal to the plane of equation (ii) and the given line are perpendicular.

$\Rightarrow (\hat{\imath} - \hat{\jmath} + 2\hat{k}) \cdot \lambda(b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}) = 0$

$\Rightarrow \lambda(b_1 - b_2 + 2b_3) = 0$

$\Rightarrow b_1 - b_2 + 2b_3 = 0 \quad \ldots\text{(iv)}$

Similarly, $(3\hat{\imath} + \hat{\jmath} + \hat{k}) \cdot \lambda(b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}) = 0$

$\Rightarrow \lambda(3b_1 + b_2 + b_3) = 0$

$\Rightarrow 3b_1 + b_2 + b_3 = 0 \quad \ldots\text{(v)}$

From equations (iv) and (v), we obtain

$$\frac{b_1}{(-1 \times 1) - (1 \times 2)} = \frac{b_2}{(2 \times 3) - (1 \times 1)} = \frac{b_3}{(1 \times 1) - (3 \times -1)}$$

$\Rightarrow \frac{b_1}{-3} = \frac{b_2}{5} = \frac{b_3}{4}$

Therefore, the direction ratios of $\vec{b}$ are $-3, 5$ and 4.

$\therefore \vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k} = -3\hat{\imath} + 5\hat{\jmath} + 4\hat{k}$

Substituting the value of $\vec{b}$ in equation (i), we get

$\vec{r} = (\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) + \lambda(-3\hat{\imath} + 5\hat{\jmath} + 4\hat{k})$

Hence, $\vec{r} = (\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) + \lambda(-3\hat{\imath} + 5\hat{\jmath} + 4\hat{k})$ is the required equation of the line.

20. Find the vector equation of the line passing through the point $(1, 2, -4)$ and perpendicular to the two lines: $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$

Solution:

Given: Two lines are $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$

Let the required line be parallel to the vector $\vec{b}$ given by $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

The position vector of the point $(1, 2, -4)$ is $\vec{a} = \hat{\imath} + 2\hat{\jmath} - 4\hat{k}$

The equation of the line passing through $(1, 2, -4)$ and parallel to vector $\vec{b}$ is

$\vec{r} = \vec{a} + \lambda\vec{b}$

$\Rightarrow \vec{r}(\hat{\imath} + 2\hat{\jmath} - 4\hat{k}) + \lambda(b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}) \quad \ldots\text{(i)}$

The equations of the lines are

$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}$ ...(ii)

$\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$ ...(iii)

Since, equation (i) and equation (ii) are perpendicular to each other,

$\therefore 3b_1-16b_2+7b_3=0$ ...(iv)

Also, equation (i) and equation (iii) are perpendicular to each other.

$\therefore 3b_1+8b_2-5b_3=0$ ...(v)

From equations (iv) and (v), we get

$$\frac{b_1}{(-16\times -5)-(8\times 7)}=\frac{b_2}{(7\times 3)-(3\times -5)}=\frac{b_3}{(3\times 8)-(3\times -16)}$$

$$\Rightarrow \frac{b_1}{24}=\frac{b_2}{36}=\frac{b_3}{72}$$

$$\Rightarrow \frac{b_1}{2}=\frac{b_2}{3}=\frac{b_3}{6}$$

$\therefore$ Direction ratios of $\vec{b}$ are $2, 3$ and 6.

$\Rightarrow \vec{b}=2\hat{\imath}+3\hat{\jmath}+6\hat{k}$

Substituting $\vec{b}=2\hat{\imath}+3\hat{\jmath}+6\hat{k}$ in equation (i), we get

$\vec{r}=(\hat{\imath}+2\hat{\jmath}-4\hat{k})+\lambda(2\hat{\imath}+3\hat{\jmath}+6\hat{k})$

Hence, $\vec{r}=(\hat{\imath}+2\hat{\jmath}-4\hat{k})+\lambda(2\hat{\imath}+3\hat{\jmath}+6\hat{k})$ is the required equation of the line.

21. Prove that if a plane has the intercepts a, b, c and is at a distance of P units from the origin, then $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}$

Solution:

Given: A plane has the intercepts a, b, c and is at a distance of P units from the origin

To prove: $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}$

Proof:

The equation of a plane having intercepts a, b, c with x, y and z axes respectively is given by,

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ ...(i)

The distance (p) of the plane from the origin is given by,

$$p=\left|\frac{\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}}\right|$$

$\Rightarrow p=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$

$$\Rightarrow p^2=\frac{1}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}$$

$\Rightarrow \frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$

Hence, it is proved.

22. Distance between the two planes: $2x+3y+4z=4$ and $4x+6y+8z=12$ is

(A) 2 units

(B) 4 units

(C) 8 units

(D) $\frac{2}{\sqrt{29}}$ units

Solution:

(D)

The given equations of the planes are

$2x+3y+4z=4$ ...(i)

$4x+6y+8z=12$

$\Rightarrow 2x+3y+4z=6$ ...(ii)

Here, the given planes are parallel.

We know that, the distance between two parallel planes $ax + by + cz = d_1$ and $ax + by + cz = d_2$ is given by

$$D = \left|\frac{d_2 - d_1}{\sqrt{a^2 + b^2 + c^2}}\right|$$

$$\Rightarrow D = \left|\frac{6 - 4}{\sqrt{(2)^2 + (3)^2 + (4)^2}}\right|$$

$$D = \frac{2}{\sqrt{29}}$$

Hence, the distance between the lines is $\frac{2}{\sqrt{29}}$ units.

Therefore, the correct answer is D.

23. The planes: $2x - y + 4z = 5$ and $5x - 2.5y + 10z = 6$ are

(A) Perpendicular

(B) Parallel

(C) Intersect y-axis

(D) Passes through $\left(0, 0, \frac{5}{4}\right)$

Solution:

(B)

The given equations of the planes are

$2x - y + 4z = 5$ ...(i)

$5x - 2.5y + 10z = 6$ ...(ii)

From the above equations we get,

$$\frac{a_1}{a_2} = \frac{2}{5}$$

$$\frac{b_1}{b_2} = \frac{-1}{-2.5} = \frac{2}{5}$$

$$\frac{c_1}{c_2} = \frac{4}{10} = \frac{2}{5}$$

$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Thus, the given planes are parallel.

Therefore, the correct answer is B.